Measuring Uncertainty about Long-Run Predictions

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Presentation at ECB

June 14, 2014

Set-up

- Observe data x_t, $t = 1, \cdots, T$, such as growth rates of GDP, or inflation

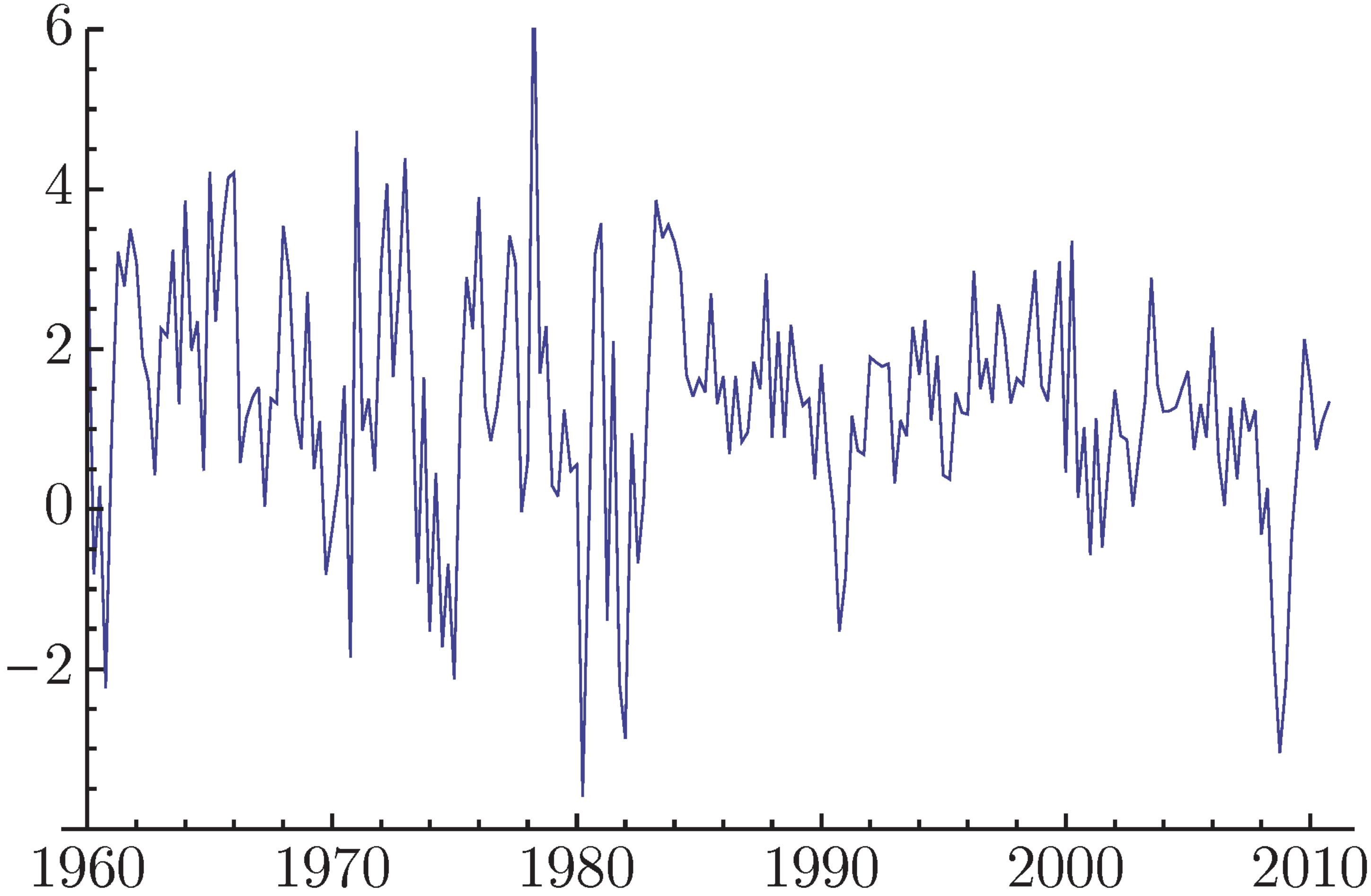
- Want to forecast *average* over the next $\lfloor \lambda T \rfloor$ periods

$$f = \lfloor \lambda T \rfloor^{-1} \sum_{l=1}^{\lfloor \lambda T \rfloor} x_{T+l}$$

  where $\lambda = 0.5$, say.

- Aim: Construct interval from data $\{x_t\}_{t=1}^{T}$ that contains f with, say, 90% probability in repeated samples

US Postwar GDP Per Capita



Focus and Challenges

- This paper: statistical (rather than "structural") univariate long-term forecasting

- Econometric challenges
  - Only limited sample information about long-term behavior
  - Set of plausible models of long-term behavior?
  - How to deal with model and/or parameter uncertainty?

Low-Frequency Transformations

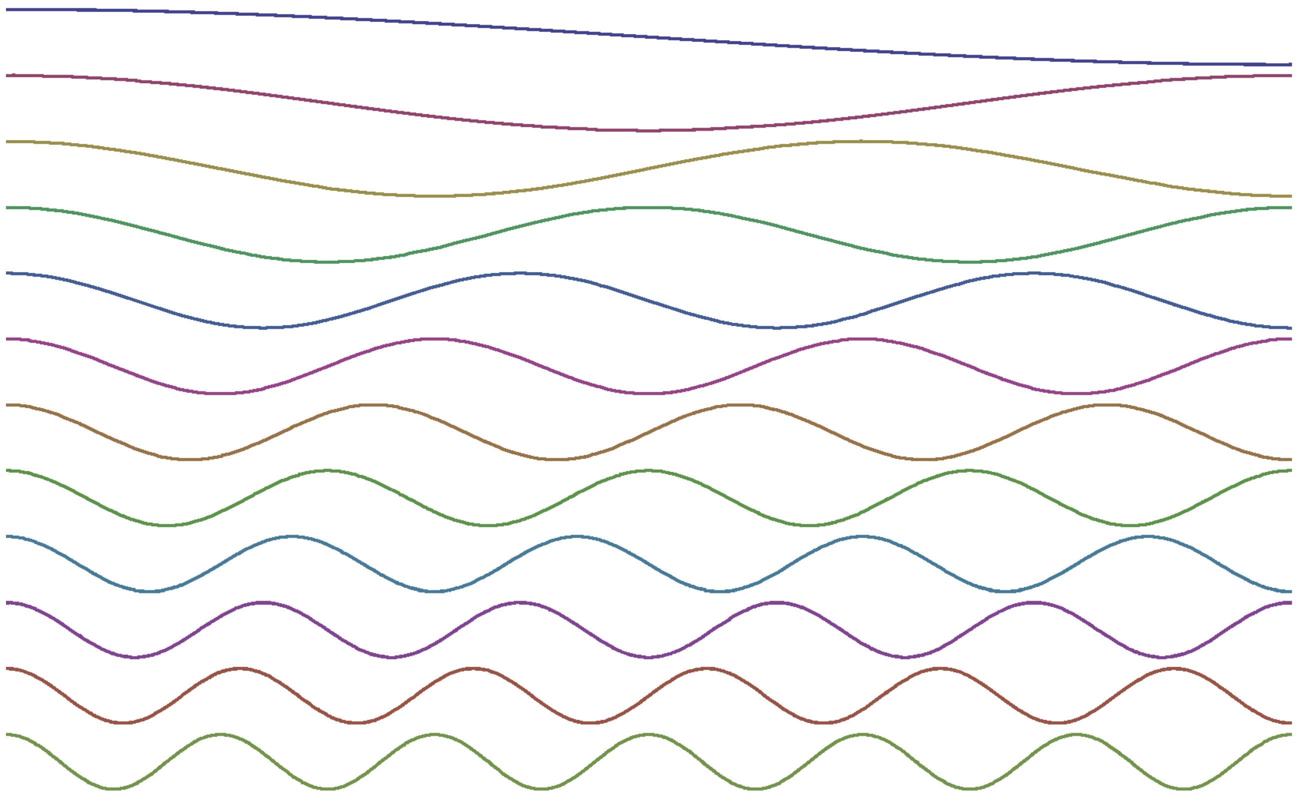
- Intuitively, question concerns low-frequency properties of x_t.

- Extract relevant information by computing low-frequency transforms (Müller and Watson, 2008)

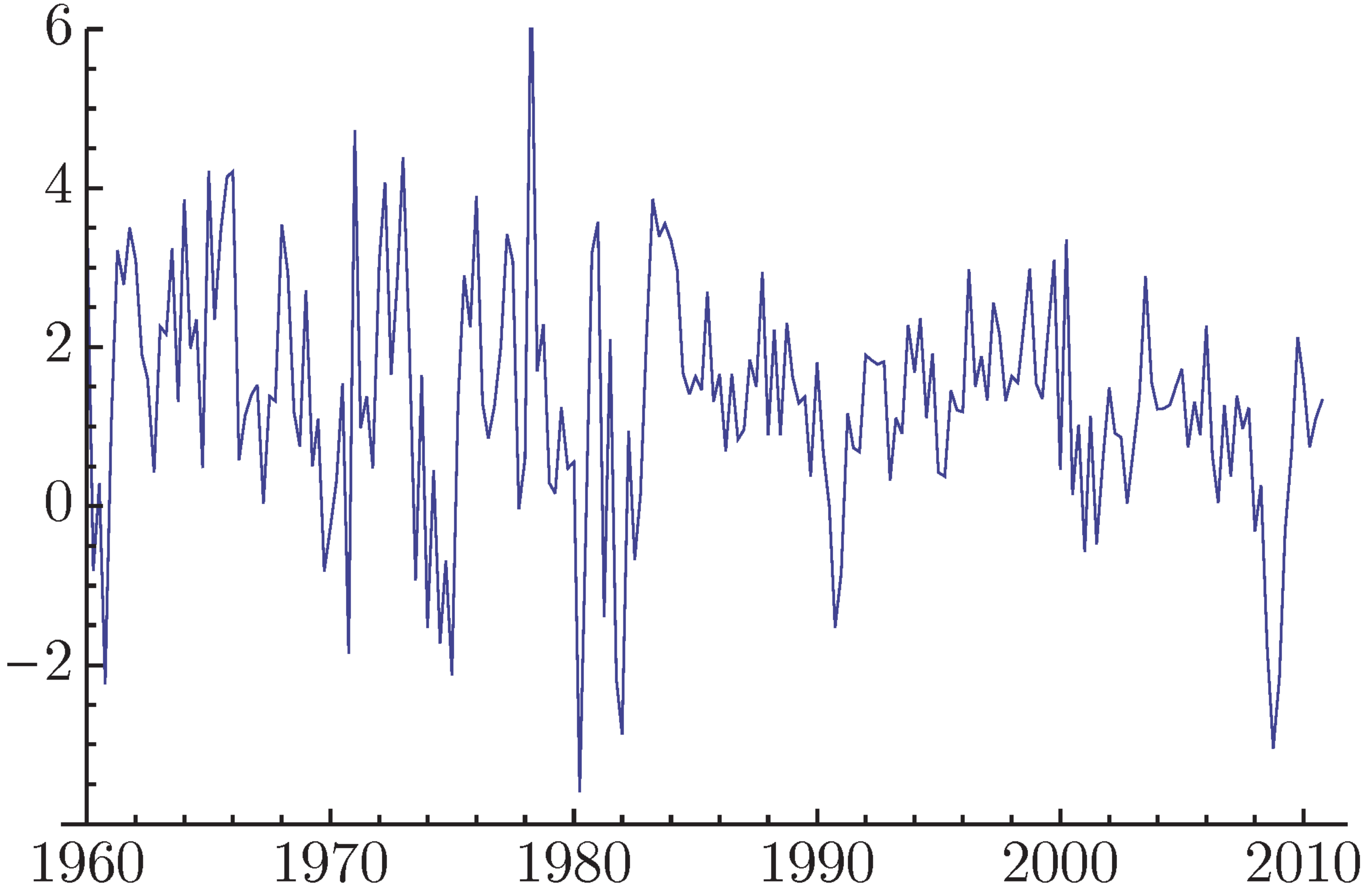
$$X_j = T^{-1} \sum_{t=1}^{T} \sqrt{2} \cos(\pi j t / T) x_t, \quad j = 1, \cdots, q$$

where q is a number like $q = 12$, and treat $(X_1, \cdots, X_q)'$ and $\hat{\mu} = T^{-1} \sum_{t=1}^{T} x_t$ as only available data

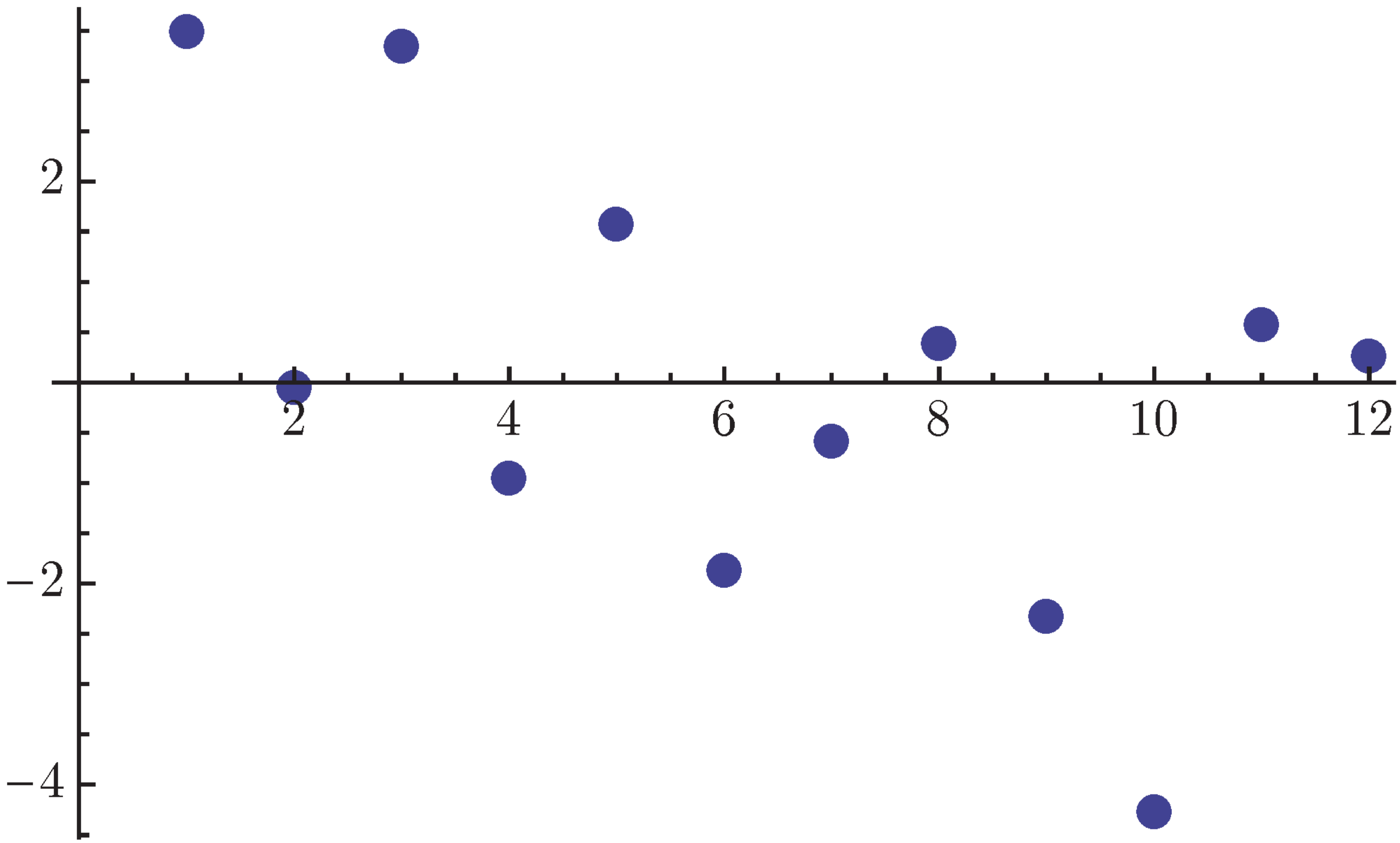
Cosine Weights



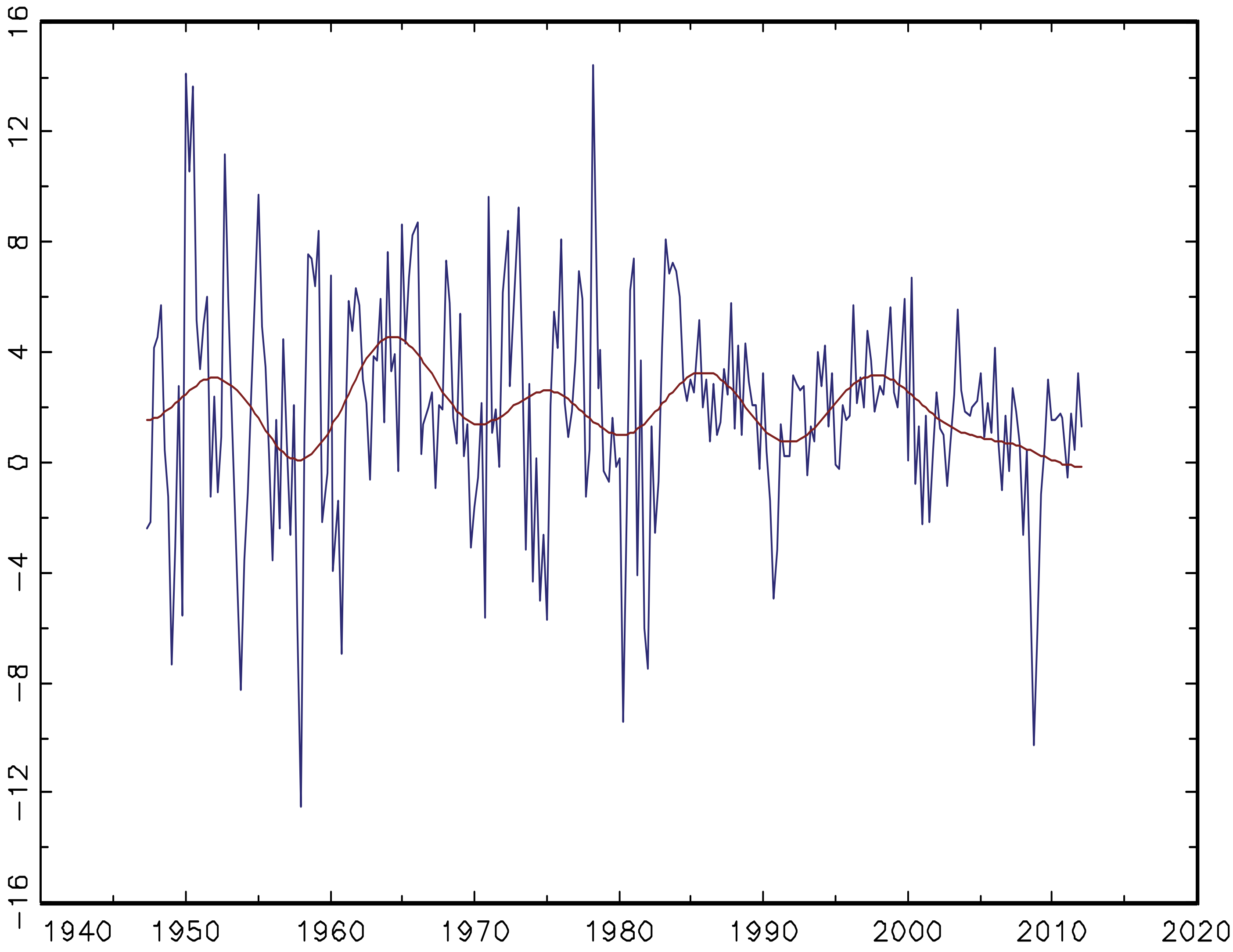
GDP



$q = 12$ LF Transforms for GDP



GDP LF Projection



Pros and Cons of LF Transforms

- Extract low-frequency information in $\{x_t\}$

- Avoids modelling and potential misspecification of higher frequency aspects

- Captures notion that relevant sample information about long-run forecasts limited

- But potential loss of efficiency (see paper)

Standard I(0) Asymptotics for Time Series

- Under a range of primitive conditions on the dependent and heterogeneous mean-zero process $\{u_t\}$, a Central Limit Theorem holds for all fractions of the sample, i.e. for all $0 \leq r_1 < r_2 \leq s_1 < s_2$,

$$\begin{pmatrix} \frac{1}{\sqrt{T}} \sum_{t=\lfloor r_1 T \rfloor + 1}^{\lfloor r_2 T \rfloor} u_t \\ \frac{1}{\sqrt{T}} \sum_{t=\lfloor s_1 T \rfloor + 1}^{\lfloor s_2 T \rfloor} u_t \end{pmatrix} \Rightarrow \mathcal{N}\left(0, \begin{pmatrix} \sigma^2(r_2 - r_1) & 0 \\ 0 & \sigma^2(s_2 - s_1) \end{pmatrix}\right)$$

- This (almost) implies the "Functional" Central Limit Theorem for nicely behaved I(0) processes

$$T^{-1/2} \sum_{t=1}^{\lfloor \cdot T \rfloor} u_t \Rightarrow \sigma W(\cdot)$$

Implications for Low-Frequency Transformations

- Suppose $x_t = \mu + u_t$ and u_t is I(0) in the sense $T^{-1/2} \sum_{t=1}^{\lfloor \cdot T \rfloor} u_t \Rightarrow \sigma W(\cdot)$

- Cosine weights are orthogonal to constant:

$$X_j = \frac{\sqrt{2}}{\sqrt{T}} \sum_{t=1}^{T} \cos(\pi j t / T) x_t = \frac{\sqrt{2}}{\sqrt{T}} \sum_{t=1}^{T} \cos(\pi j t / T) u_t$$

- With $\hat{\mu} = T^{-1} \sum_{t=1}^{T} x_t$, $f = \lfloor \lambda T \rfloor^{-1} \sum_{l=1}^{\lfloor \lambda T \rfloor} x_{T+l}$, $X_0 = \sqrt{T}(\hat{\mu} - \mu)$, and $Y = \sqrt{T}(f - \mu)$, we obtain

$$(X_0, \cdots, X_q, Y)' \Rightarrow \mathcal{N}(0, \sigma^2 \mathbf{\Sigma})$$

since weighted averages of Gaussian processes are multivariate Gaussian

- If μ and $\sigma^2 \mathbf{\Sigma}$ were known, then could simply report 90% set of the (suitably scaled and centered) conditionally normal distribution $Y | \{X_j\}_{j=0}^{q}$

Invariance

- Impose scale and translation invariance:

$$\{x_t\}_{t=1}^T \mapsto \{m + cx_t\}_{t=1}^T \quad \text{for any } m \text{ and } c \neq 0$$

must lead to corresponding transformation of predictive set

- Can show: Under invariance, asymptotic problem becomes construction of prediction set of

$$Y^s = \frac{Y - X_0}{s_X} \text{ given } X^s = \left(\frac{X_1}{s_X}, \cdots, \frac{X_q}{s_X} \right)'$$

where $s_X^2 = q^{-1} \sum_{j=1}^q X_j^2$

$\Rightarrow$ Invariance takes care of lack of knowledge of μ and σ (but still need to know $\mathbf{\Sigma}$ to compute the conditional distribution)

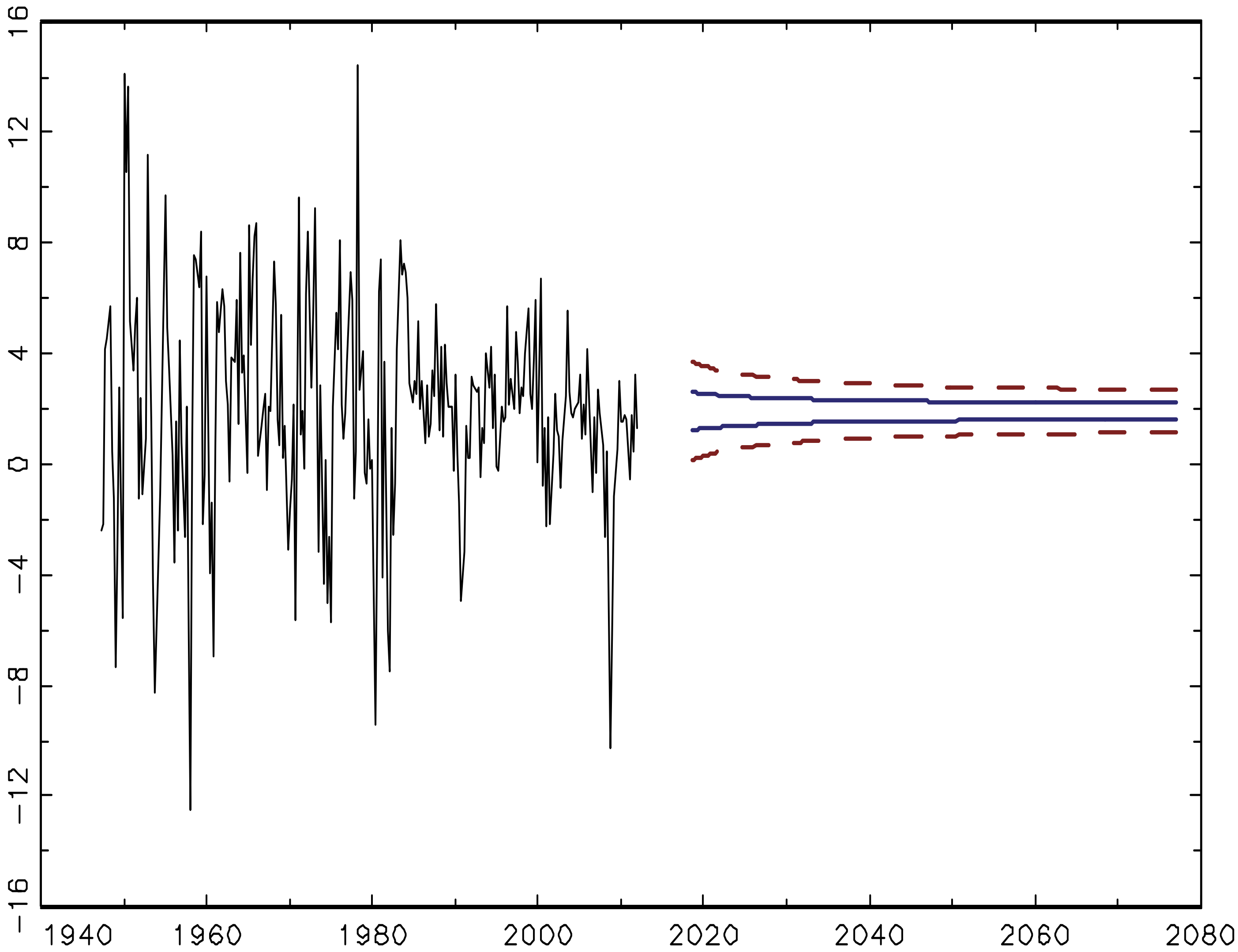
Low-Frequency Forecasts—I(0) Model

- In I(0) model, it turns out that
  - $Y^s = \frac{Y - X_0}{s_X}$ is scaled Student-t$_q$, scaled by $\sqrt{1 + \lambda^{-1}}$
  - X^s is independent of Y^s

$\Rightarrow$ intervals for f are of the form $\hat{\mu} \pm$ student-t quantiles multiplied by

$$\frac{(1 + \lambda^{-1})^{1/2} s_X}{\sqrt{T}}$$

GDP 50% and 90% Intervals in I(0) Model



Beyond the I(0) Model

- Natural concern that I(0) model is "too stationary"

- Assume local-level model

$$x_t = \mu + \frac{g}{T} \sum_{s=1}^{t} \eta_s + \varepsilon_t = \mu + u_t$$

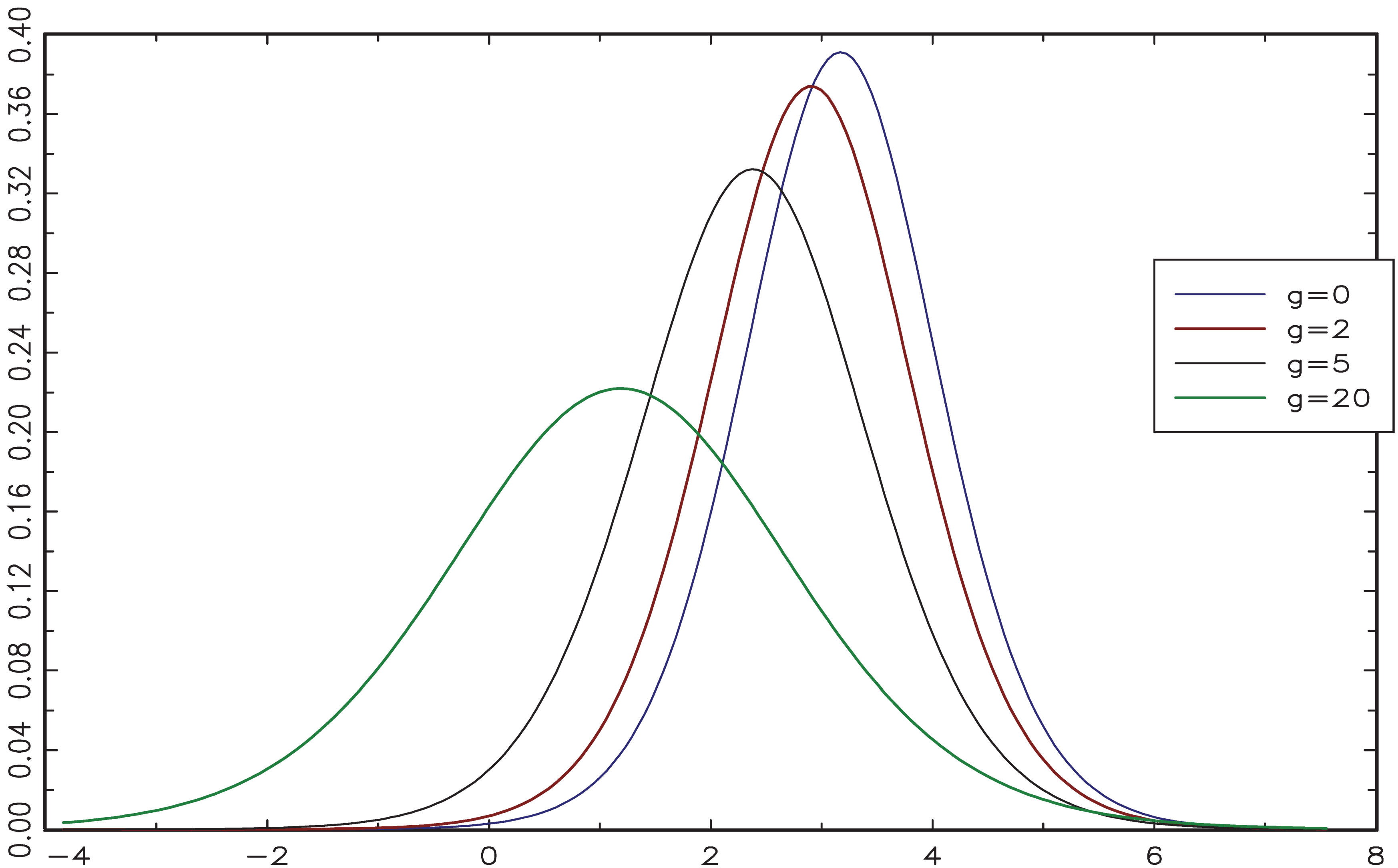
  where $\{\varepsilon_t\}$ and $\{\eta_t\}$ are I(0) with identical long-run variance σ^2, so that $g \geq 0$ measures extent of local mean variability

- Still implies

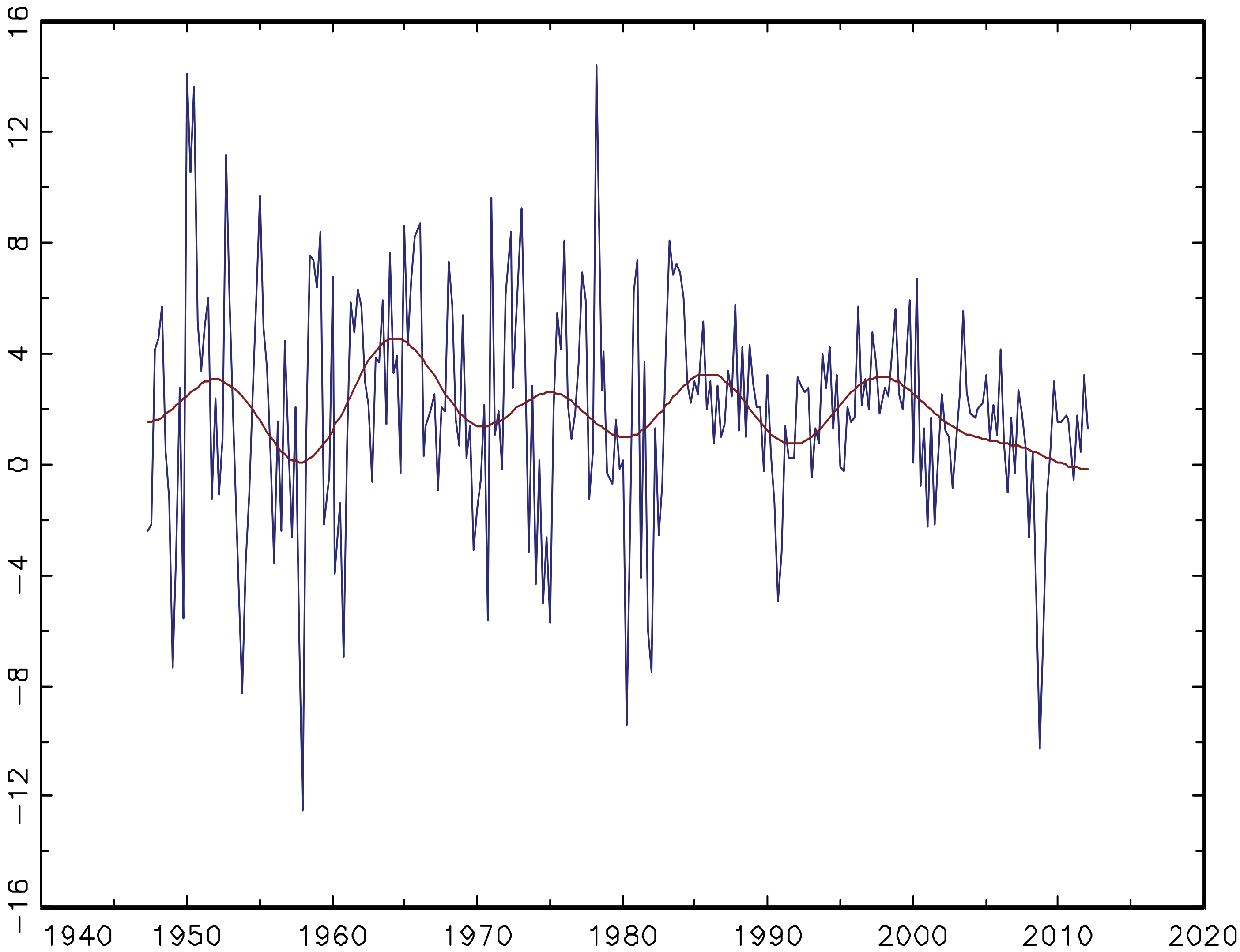
$$T^{-1/2} \sum_{t=1}^{\lfloor \cdot T \rfloor} u_t \Rightarrow \sigma G(\cdot) \tag{1}$$

  for Gaussian process G, so that $(X_0, \cdots, X_q, Y)' \Rightarrow \mathcal{N}(0, \sigma^2 \mathbf{\Sigma})$, where now $\mathbf{\Sigma} = \mathbf{\Sigma}(g)$

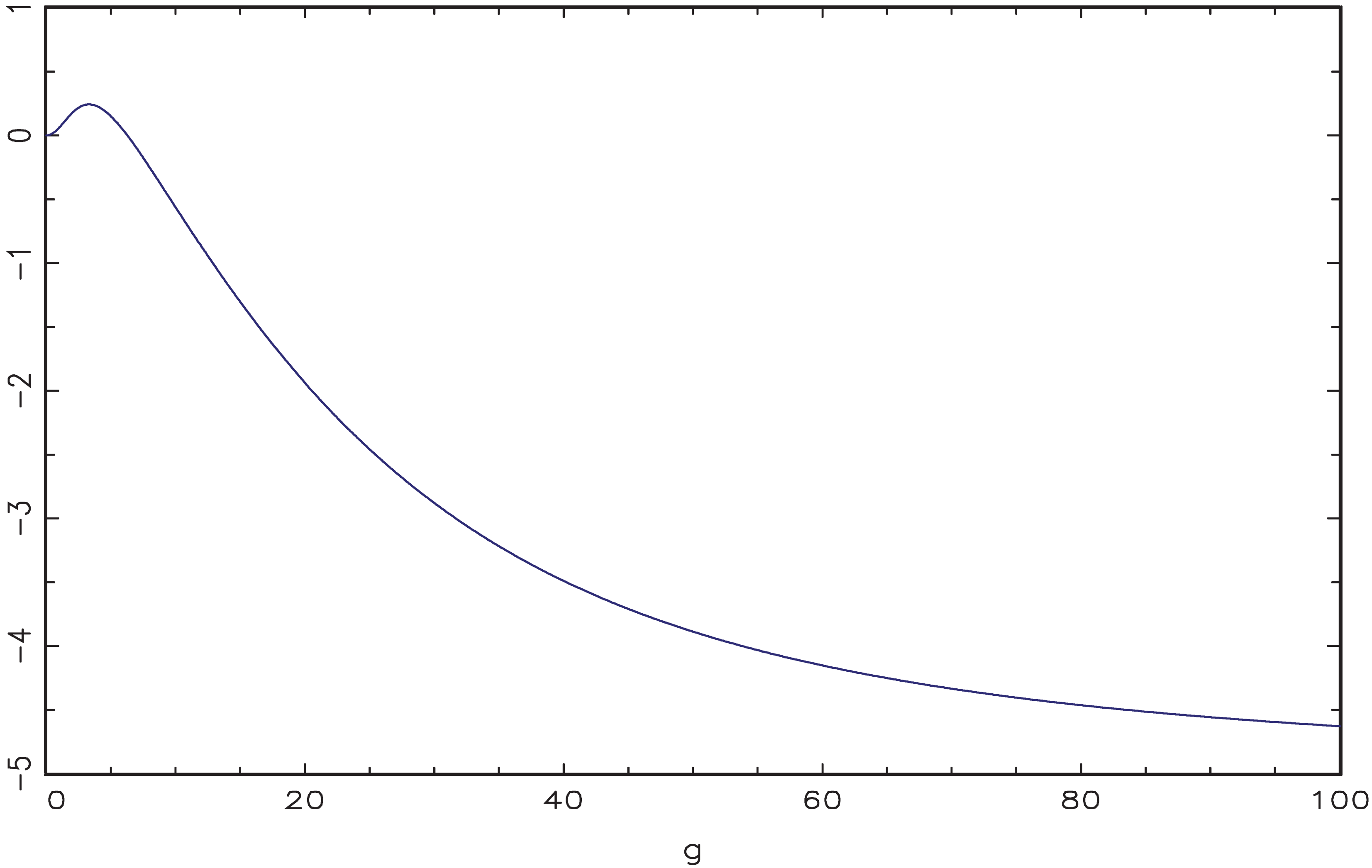
GDP Predictive Densities, LLM, $\lambda = 0.2$



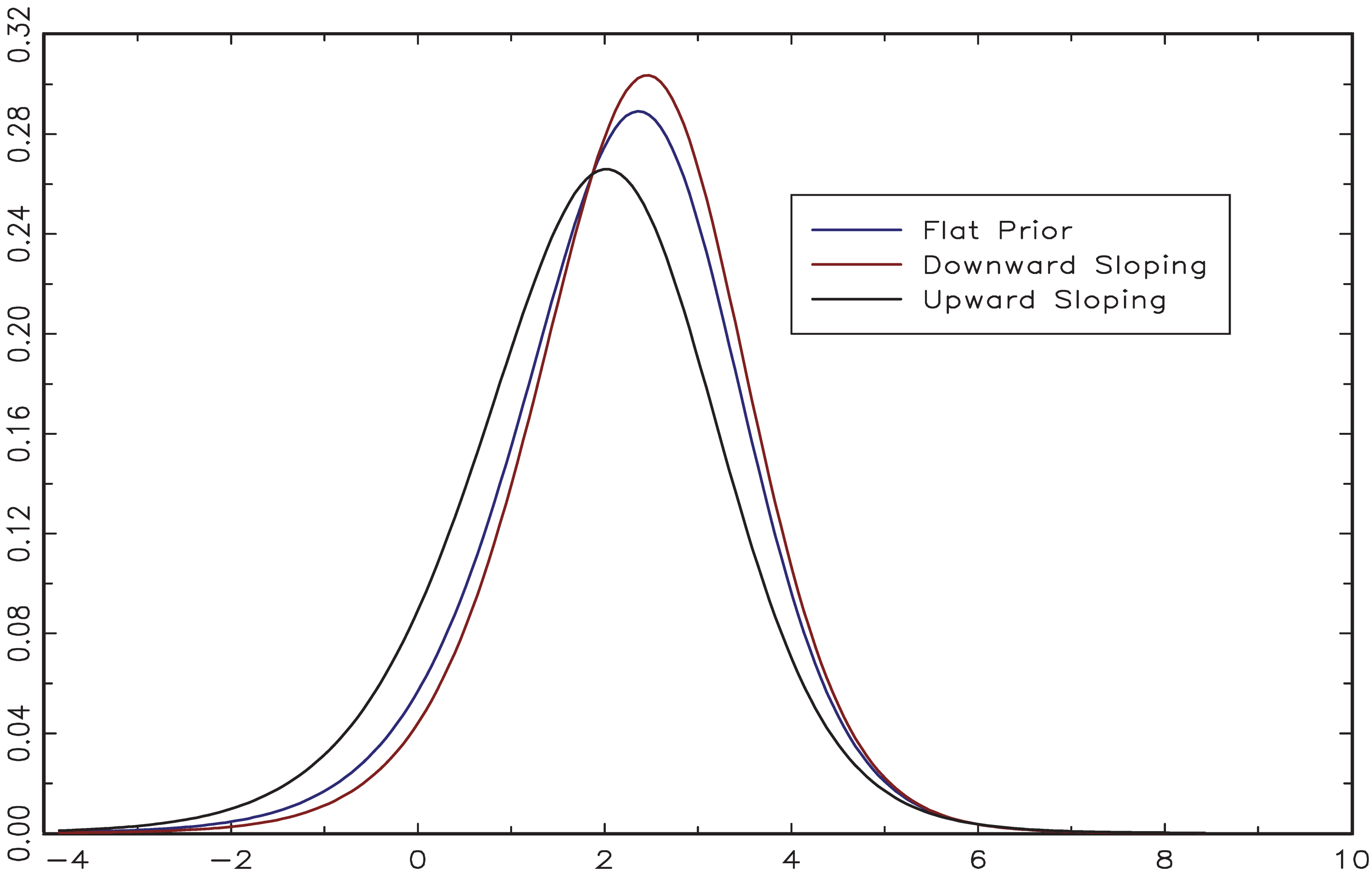
GDP LF Projection



GDP LF Likelihood in LLM



GDP Bayes Predictive Densities



Beyond the Local-Level Model

- Approach generalizes to any model $x_t = \mu + u_t$ that satisfies

$$T^{-\alpha} \sum_{t=1}^{\lfloor \cdot T \rfloor} u_t \Rightarrow \sigma G(\cdot)$$

  for some Gaussian process G and α (for example: fractional model).

- Possible to derive predictive set that remains valid for arbitrary G? No, since $\boldsymbol{\Sigma}$ then entirely unconstrained.

- Need some regularity of x_t to be able to forecast.

- Consider covariance stationarity of Δx_t (allowing mean growth rate to vary stochastically).

Local-To-Zero Spectrum

- Let $s_T : [-\pi, \pi] \mapsto \mathbb{R}_+$ be a sequence of (pseudo) spectral densities of $\{x_t\}$, and define the *local-to-zero spectrum* $S : \mathbb{R} \mapsto \mathbb{R}$ via

$$S(\omega) = T^{-2\alpha+1} \lim_{T \to \infty} s_T(\omega/T). \tag{2}$$

  for suitable α.

- Under some linear process conditions on Δx_t and (1), we show

$$(X_0, \cdots, X_q, Y)' \Rightarrow \mathcal{N}(0, \sigma^2 \mathbf{\Sigma})$$

  where $\mathbf{\Sigma}$ is a function of S.

  $\Rightarrow$ Long-run forecasting uncertainty is fully determined by (pseudo) spectral shape close to origin.

Local Spectra

- Fractional model:

$$S_d(\omega) \propto |\omega|^{-2d}, d \in (-0.5, 1.5)$$

- Local-to-Unity model (AR(1) with $\rho_T = 1 - c/T$):

$$S_c(\omega) \propto 1/(\omega^2 + c^2)$$

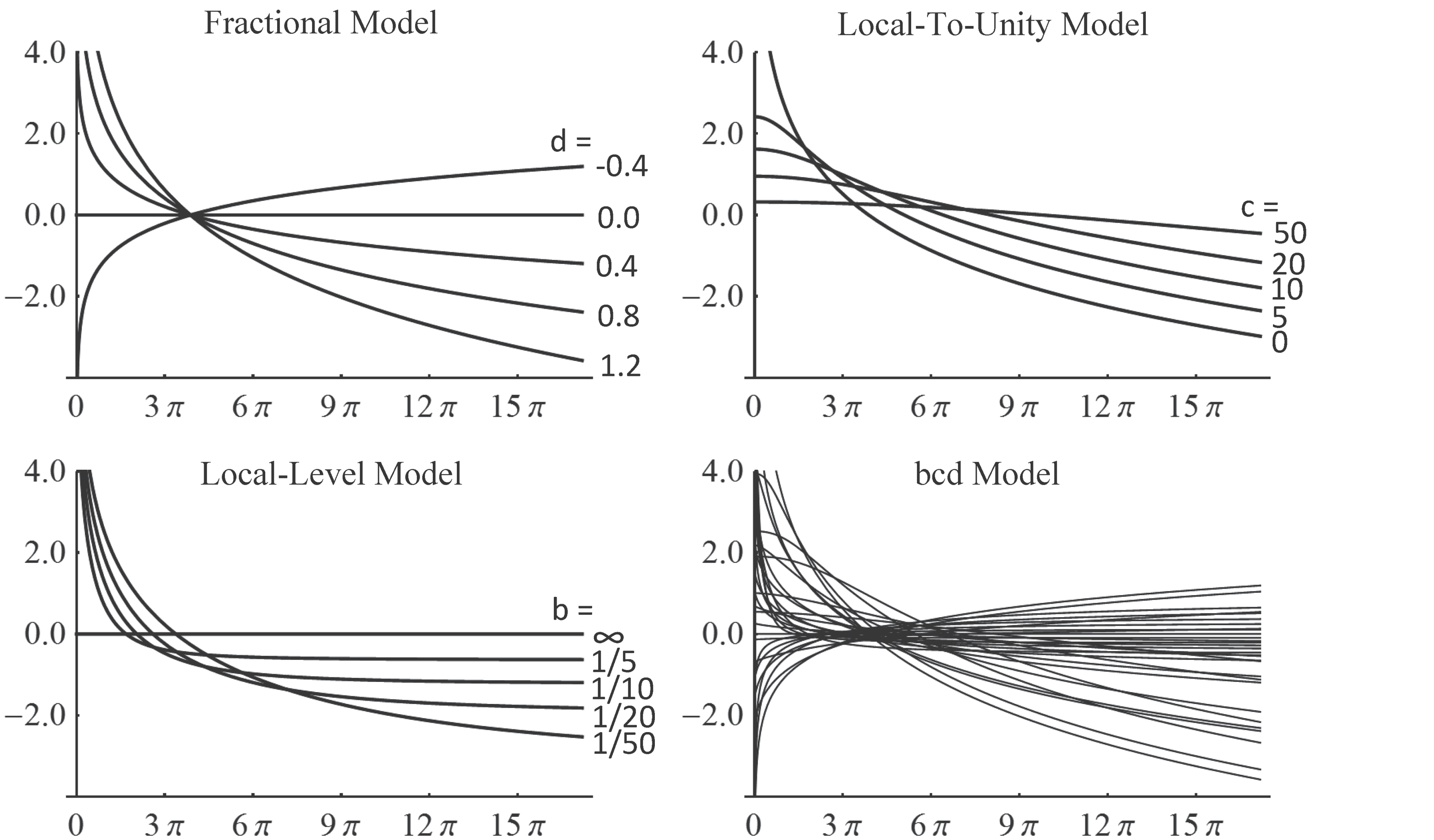
- Local-Level model $x_t = \mu + b\varepsilon_t + \frac{1}{T}\sum_{s=1}^{t} \eta_s$:

$$S_b(\omega) \propto 1/\omega^2 + b^2$$

$\Rightarrow$ All these local spectra are in bcd-family

$$S_\theta(\omega) \propto \left(\frac{1}{\omega^2 + c^2}\right)^{2d} + b^2, \quad \theta = (b, c, d)$$

Local Log-Spectra



Parameter Uncertainty

- Local spectrum depends on $\theta = (b, c, d)$, which cannot be estimated consistently by fixed number q of cosine transforms.

- Recall that via invariance, (asymptotic) problem is to forecast $Y^s = \frac{Y - X_0}{s_X}$ by $X^s = \left(\frac{X_1}{s_X}, \cdots, \frac{X_q}{s_X}\right)'$, where $s_X^2 = q^{-1} \sum_{j=1}^{q} X_j^2$.

- Let $\Psi(X^s)$ be a predictive interval of level $1 - \alpha$. Determine Ψ^* that minimizes weighted average expected length over θ, subject to coverage constraint for all values of θ:

$$\min_{\Psi} \int w(\theta) E_\theta[\text{length}(\Psi(X^s))] d\theta \quad \text{s.t.} \quad P_\theta(Y^s \in \Psi(X^s)) \geq 1 - \alpha \; \forall \, \theta \in \Theta$$

$\Rightarrow$ Almost same problem as in Müller, Elliott and Watson (2013)

Parameter Uncertainty: Conditional Properties

- Potential problem: $\Psi^*(X^s)$ could be empty for some X^s, and have otherwise unreasonable conditional properties

  $\Rightarrow$ generic potential problem of descriptions of uncertainty in nonstandard problems with sets that (only) satisfy confidence type property

  $\Rightarrow$ see Müller and Norets (2012)

- Solution: Impose that $\Psi^*(X^s)$ contains the $1 - a$ credible set relative to the prior w.

Implementation

- Set $q = 12$.

- Choose weighting function w uniformly distributed on $d \in [-0.4, 1.4]$ in fractional model

  $\Rightarrow$ seek to minimizes expected length on average with data drawn from fractional model, subject to including the $1-\alpha$ credible set with that prior and model

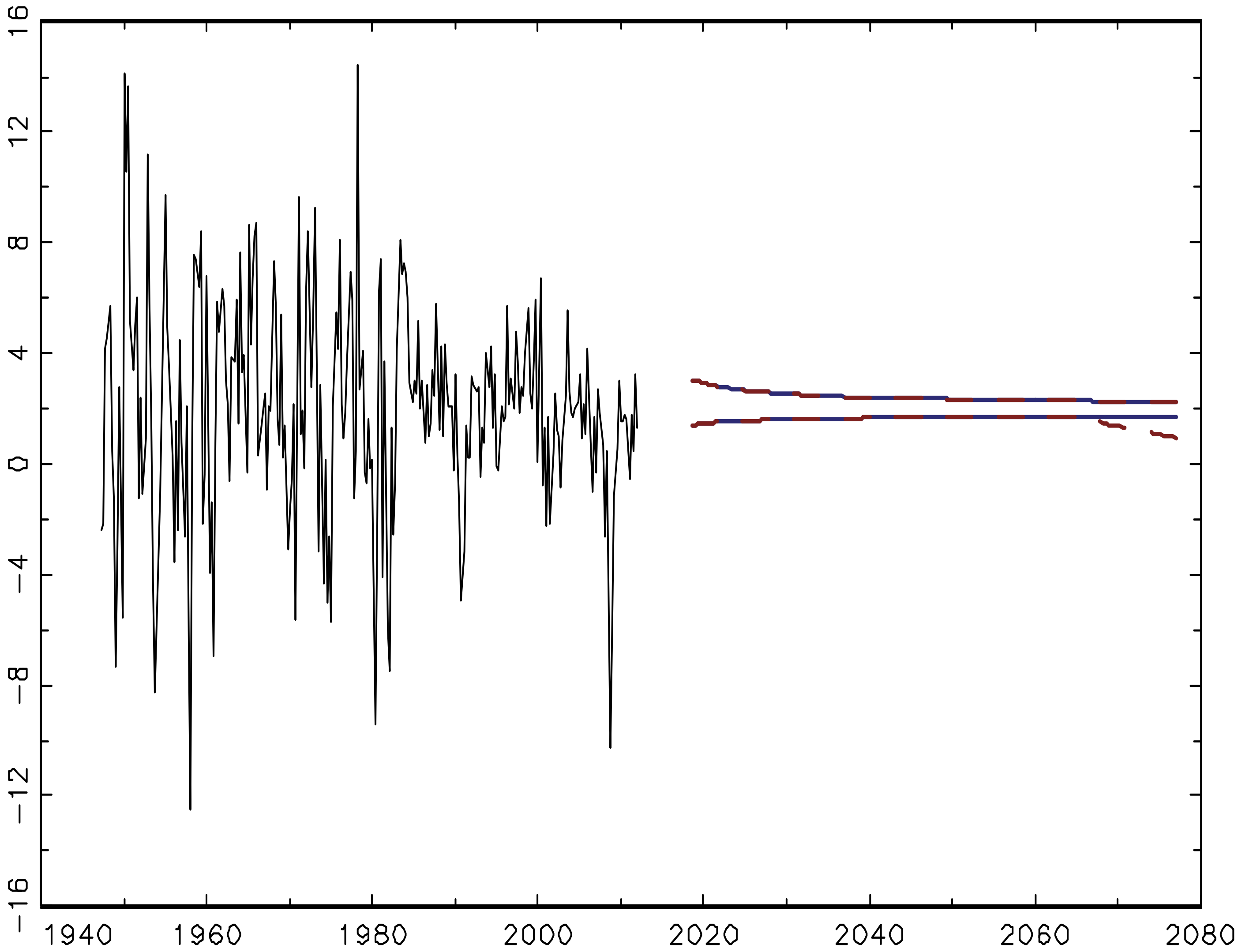
- Impose coverage $P_\theta(Y^s \in \Psi(X^s)) \geq 1-\alpha$ in larger class with local-to-zero spectrum

$$S_\theta(\omega) \propto \left(\frac{1}{\omega^2 + c^2}\right)^{2d} + b^2$$

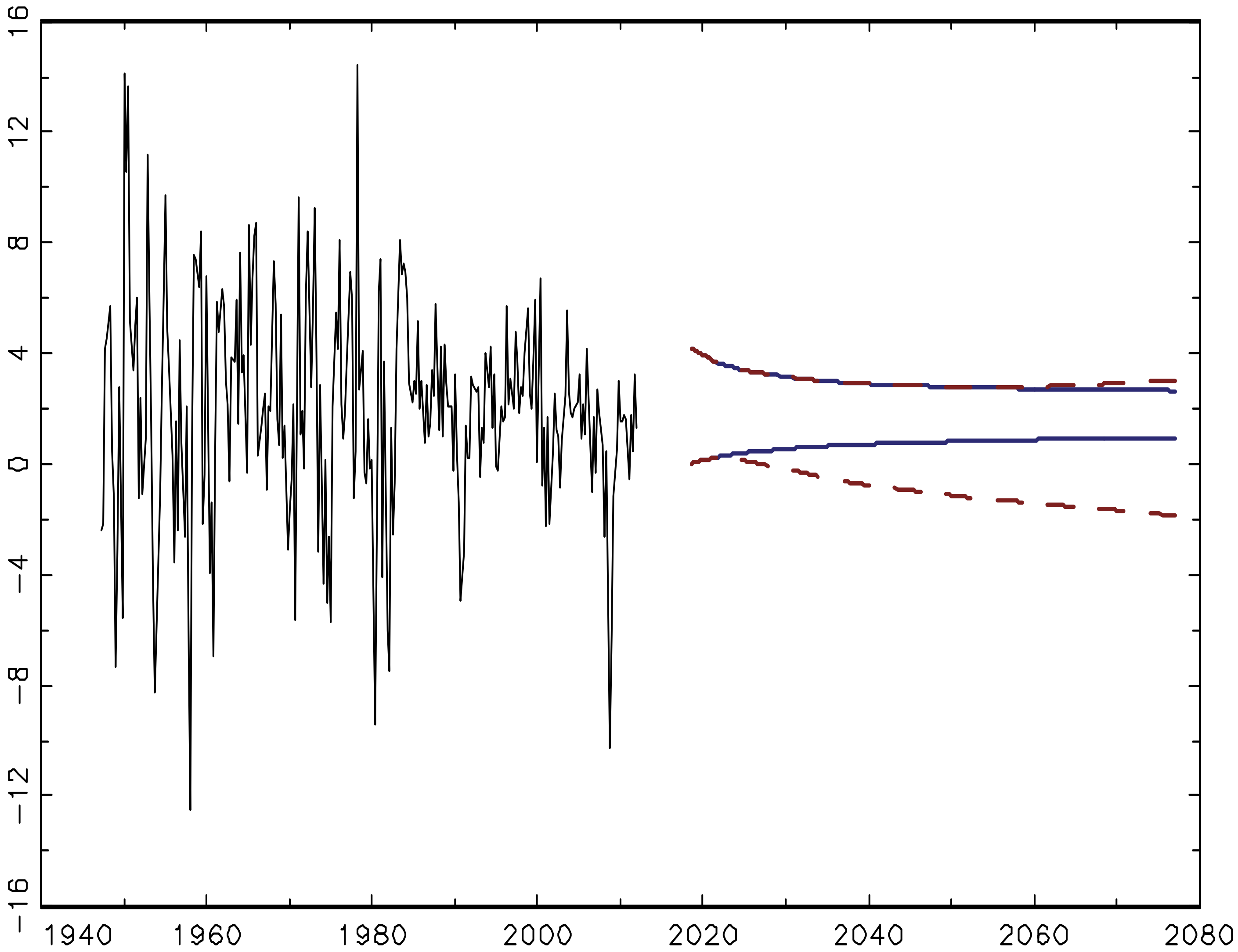
  with $d \in [-0.4, 1.4]$ and b, c arbitrary.

  $\Rightarrow$ Frequentist robustification of Bayes credible set

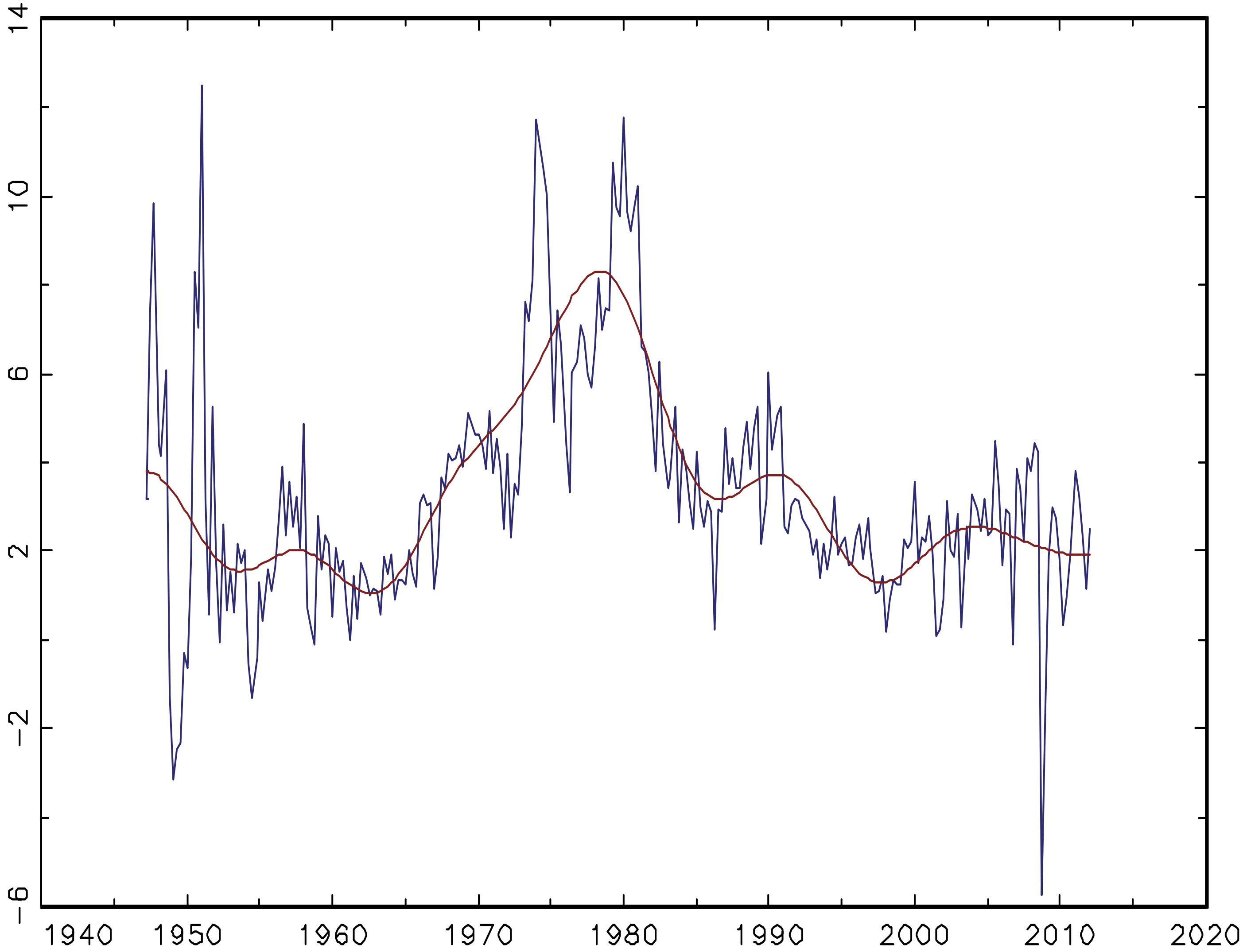
GDP 50% Interval



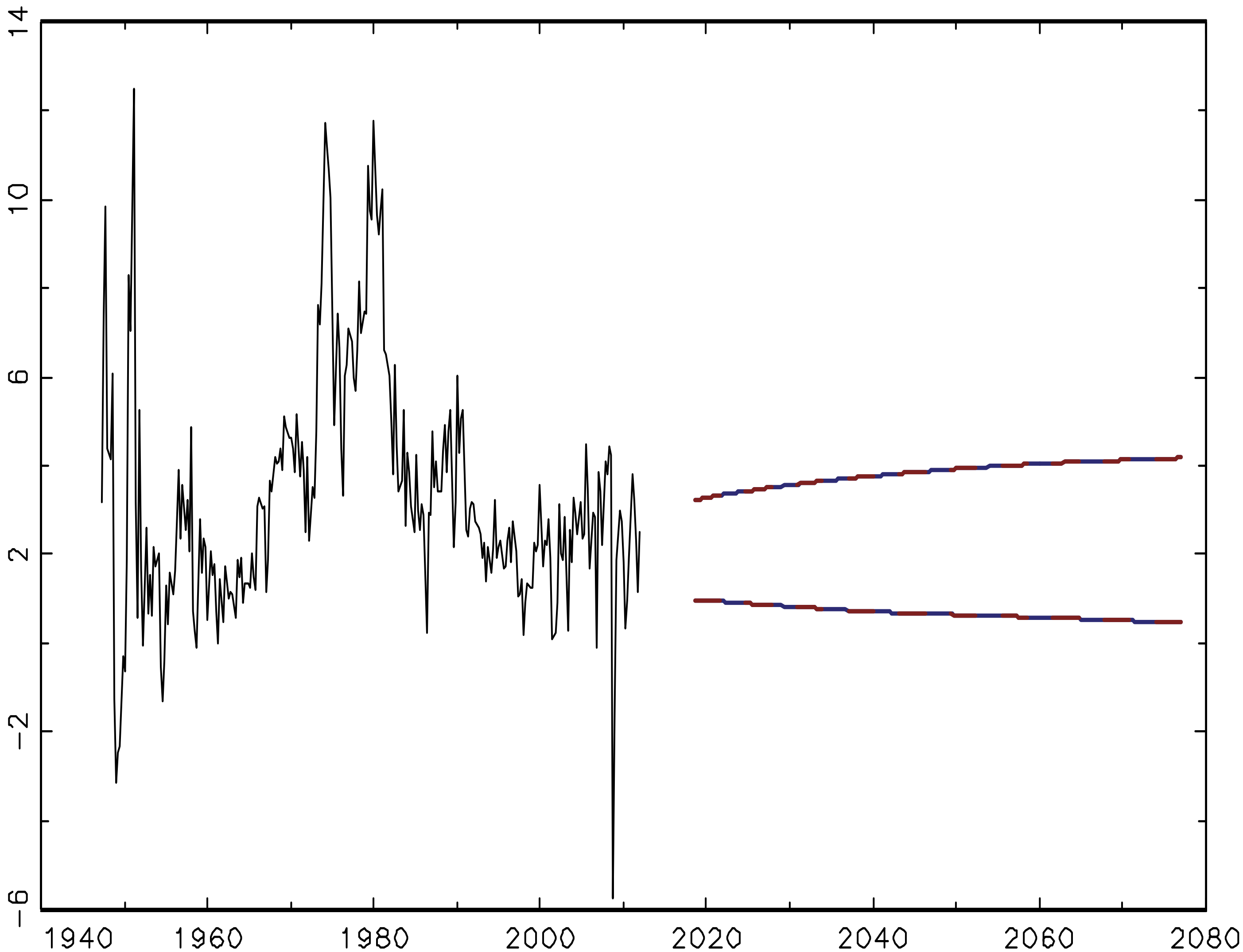
GDP 90% Intervals



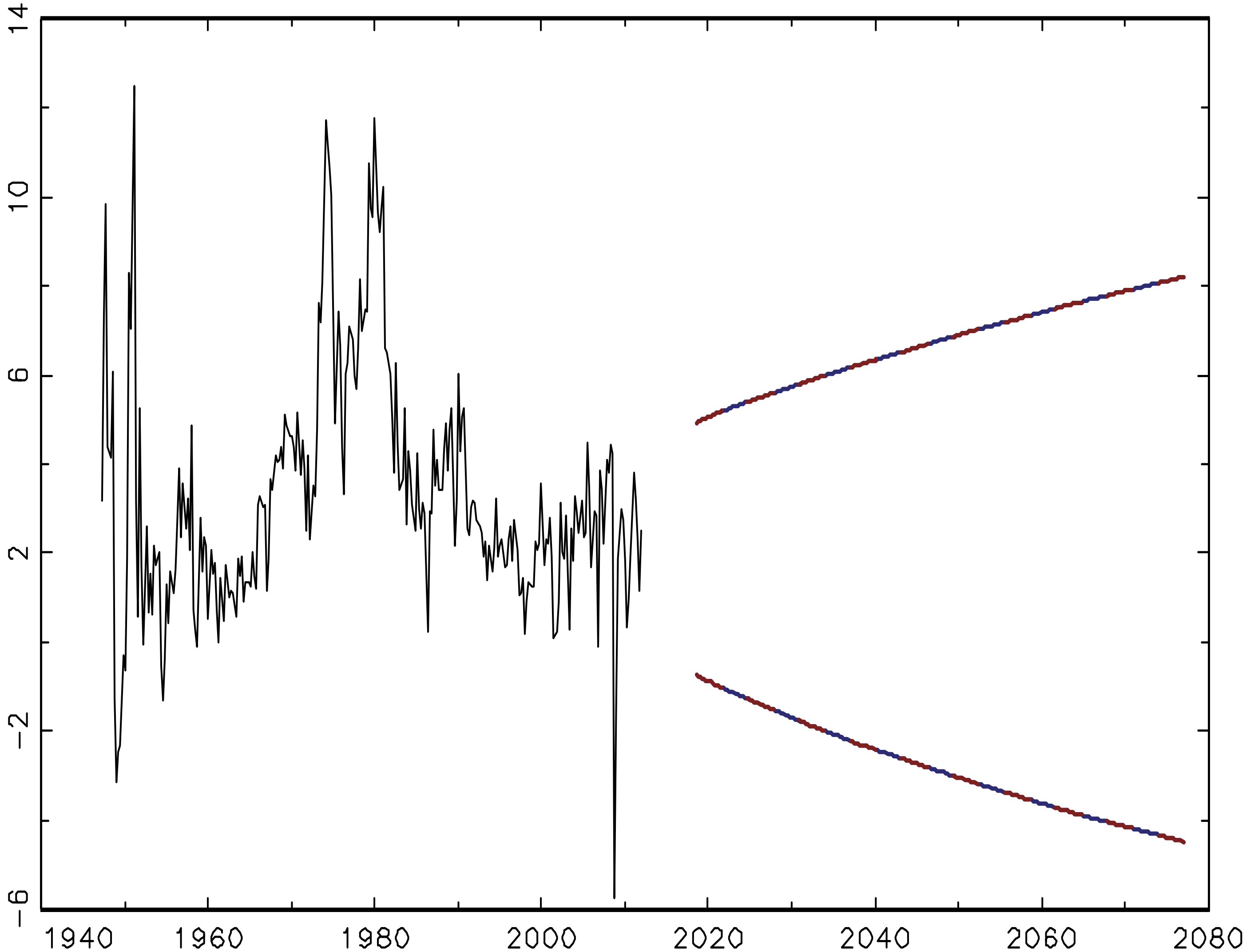
US Postwar PCE Inflation



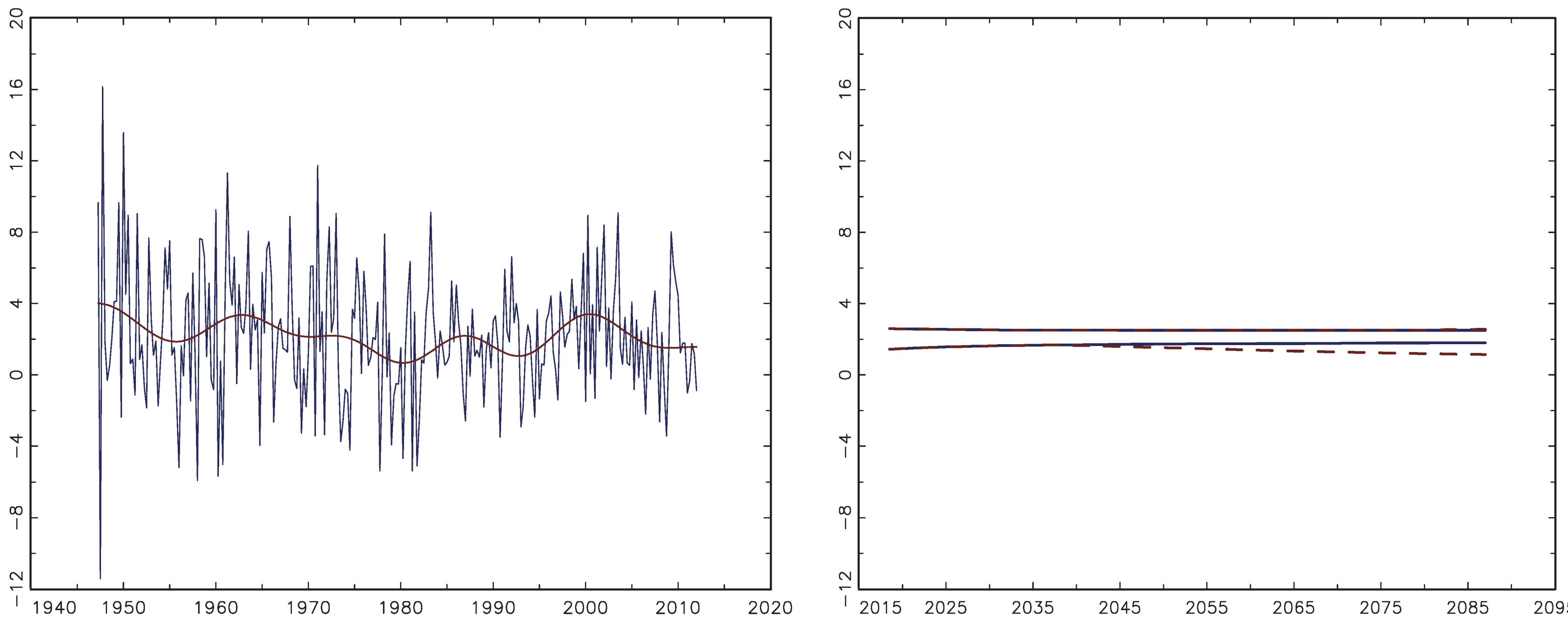
Inflation 50% Intervals



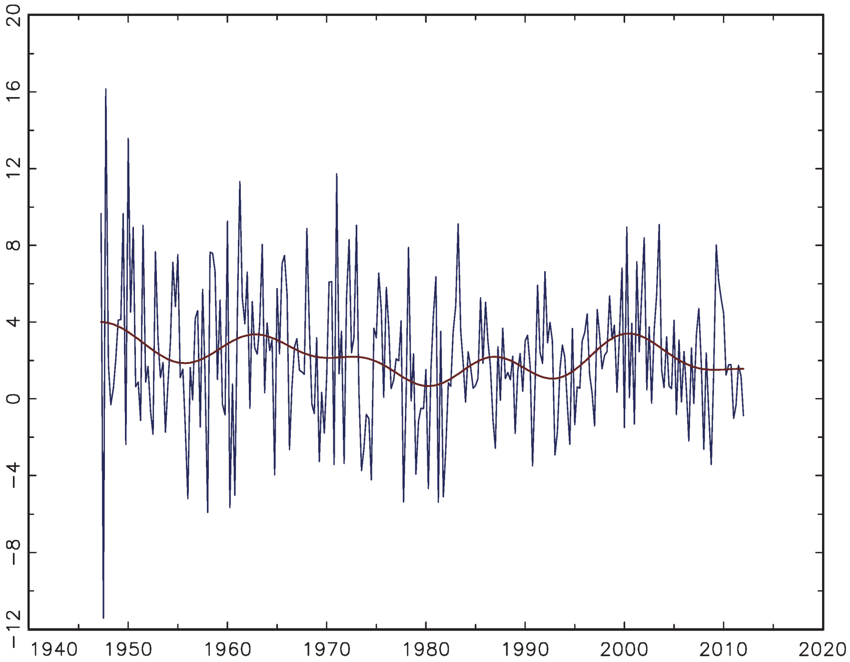
Inflation 90% Intervals

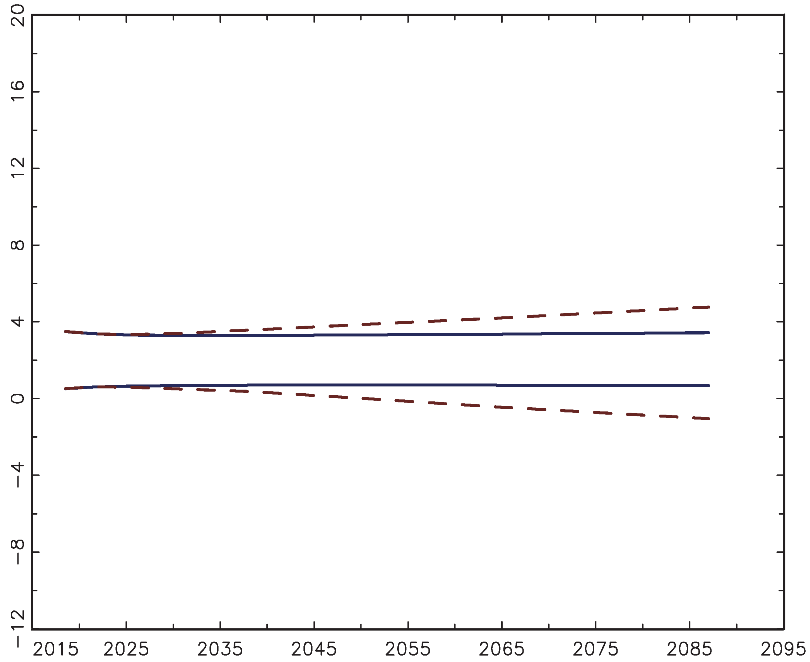


Labor Productivity 50% Interval



Labor Productivity 90% Interval





Conclusions

- Formalization of uncertainty of statistical long-term predictions
  - Low-frequency transformations to yield robustness.
  - Need regularity. Express regularity via shapes of local-to-zero spectrum.
  - Parameter uncertainty resolved by length minimizing robustification of Bayes credible sets.
- Extension to multivariate problem computationally difficult