Predictable Forecast Errors in Full-Information Rational Expectations Models with Regime Shifts*

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Abstract

This paper shows that Markov regime shifts in Full Information Rational Expectations (FIRE) models lead to predictable, regime-dependent forecast errors. This implies that ex-post forecast error regressions on current information display waves of over- and under-reaction across rolling window samples as the sequence of realized regimes changes. Using survey-based forecast data of macroeconomic aggregates, we confirm the existence of such waves. We then propose a formal econometric test that is robust to regime shifts conditional on a given data-generating process. We apply the test to a medium-scale FIRE model with regime shifts in the aggressiveness of monetary policy that is estimated on U.S. data. The model provides a close fit of observed macroeconomic dynamics and – despite the assumption of FIRE – does not allow us to decisively reject the null that the forecast error regression estimates observed in the data were generated from the model. Hence, predictability of ex-post forecast errors is, by itself, neither a sufficient condition against FIRE nor an informative metric to test alternative theories of expectations formation.

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1 Introduction

Much of modern macroeconomic research operates under the assumption that agents perfectly know the current state of the economy and form expectations rationally based on a "model-consistent" calculation of the equilibrium. One of the hallmarks of this full-information rational expectations (FIRE) hypothesis is that forecast errors are unpredictable. Yet, a growing body of research based on survey expectations data shows that ex-post, forecast errors are often predictable in systematic and quantitatively important ways. This has been taken as evidence against FIRE and has sparked a burgeoning literature introducing information frictions, departures from rational expectations, or some combination thereof to explain observed forecast error patterns.[1]

In this paper, we study the predictability of ex-post forecast errors in FIRE models in the presence of regime shifts in either model parameters or stochastic processes. Such regime shifts, due for example to changes in the economic environment or the stance of fiscal and monetary policy, are well-documented and the focus of a large literature.[2]

The main result of our investigation is that regime shifts in FIRE models lead to predictable, regime-dependent forecast errors. Intuitively, regime shifts introduce uncertainty about the future probability distribution of variables. Agents incorporate this uncertainty by forming expectations as a weighted average of regime-conditional forecasts. Forecast errors, measured ex-post after a particular regime has realized, are therefore systematically related to information available at the time of forecast.

The result has two important implications. First, in the presence of regime shifts, a researcher estimating reduced-form regressions of ex-post forecast errors on current information may find significant non-zero coefficients even if the data has been generated under FIRE. The sign of the estimated coefficient depends on the sample sequence of realized regimes relative to agents' expectations. Hence, regime shifts produce waves of over- and under-reaction of expectations to current information across rolling window regressions as the sample sequence of regime realizations changes. Forecast error predictability vanishes only as the sample grows large and the distribution of regime realizations converges to its population counterpart (and thus agents' expectations). In

[1] See Coibion and Gorodnichenko (2015), Angeletos et al. (2020), Bordalo et al. (2020), Kohlhas and Walther (2021), or Farmer et al. (2023) among many others. Also see Coibion et al. (2018) for a summary of the literature.

[2] Prominent examples include Clarida et al. (2000), Leeper and Zha (2003), Stock and Watson (2003), Cogley and Sargent (2004), Lubik and Schorfheide (2004), Boivin and Giannoni (2006), Sims and Zha (2006) or Bianchi (2013). Also see Hamilton (2016) for a survey and references therein.

the limit, unpredictability of forecast errors therefore remains a hallmark of FIRE even in the presence of regime shifts. But regime shifts may be too infrequent for this convergence to occur in available samples of macroeconomic forecasting data.

Second, in the presence of regime shifts, the coefficients of forecast error regressions on their own do not have a structural interpretation and are therefore not informative about the underlying expectations formation process. This is because ex-post forecast errors by forward-looking agents – whether fully informed rational or not – are a complicated function of the sample sequence of regime realization that are generally unobserved by the researcher. In addition, we show that with the exception of stylized examples, the variables used as predictors in forecast error regressions do not span the information set that agents use. Hence, the regressions are likely to be subject to omitted variable bias. We view this implication as perhaps most important since the literature has used estimates from forecast error regressions to argue in favor or against specific forms of information frictions or departures from rationality.

In summary, the upshot of the paper is that predictability of forecast errors is not a sufficient condition to reject FIRE and that the estimates from forecast error regressions on their own do not provide guidance about alternative theories of expectations formation. This should be taken as neither an endorsement of FIRE nor a dismissal of alternative theories of expectations formation. Indeed, there is much empirical evidence that even relatively sophisticated market participants are subject to imperfect information and make decisions that are hard to square with the assumption of rational expectations.[3] Furthermore, there is pervasive evidence of heterogeneity in the level and accuracy of forecasts across economic agents, directly contradicting the FIRE hypothesis.[4] Instead, the question is whether FIRE constitutes an appropriate metaphor to model average expectations and aggregate fluctuations, or whether alternative theories of expectations formation provide a superior approach. Our results suggest that to answer this question, researchers should test alternative theories of expectations formation against FIRE as part of structural equilibrium models that incorporate plausible regime shifts. Such an evaluation may include data on expectations, as advocated by Coibion et al. (2018), and may include moments from reduced-form forecast error regressions for candidate models to match.

The rest of the paper proceeds as follows. Section 2 sets the stage by reviewing the existing

[3]See for instance Tversky and Kahneman (1973), Tversky and Kahneman (1974), Kahneman and Tversky (1973), De Bondt and Thaler (1985), De Bondt and Thaler (1989), Adam (2007), Malmendier and Nagel (2016), Afrouzi et al. (2020), among many others.

[4]See Carroll (2003); Coibion et al. (2018); Broer et al. (2021); or Weber et al. (2022) for examples.

empirical evidence on the predictability of ex-post forecast errors with data for U.S. inflation and output growth from the Survey of Professional Forecasters (SPF). We then document that this data features waves of over- and under-reaction of forecasts to current information over rolling sample windows.

Motivated by these findings, Section 3 illustrates the implications of regime shifts for forecast error predictability in a univariate FIRE model whose coefficients switch according to a Markov process. Agents have perfect information about the current state of the economy, including the realized regime, and form rational expectations about the future based on full knowledge of the environment. The simplicity of the model admits a closed-form solution of ex-post forecast errors as a function of the current state, with the sign of this relation depending on the future regime realization. We then derive the expected forecast error regression coefficient and show that the sign and magnitude of the estimates depend on the sequence of regime realizations over the sample period relative to agents' expectations. Hence, consistent with the empirical evidence, we should expect waves of over- and under-reaction to current information across rolling windows as the sequence of regime realizations changes. We illustrate with Monte Carlo simulations that within the context of this simple data-generating process, these waves can be sizable and that convergence of regime realizations to the unconditional distribution is slow, exceeding the available time series of survey expectations of macro aggregates.

To move beyond a simple critique of forecast error regressions, Section 4 proposes a regime-robust econometric test of FIRE. The test consists of first building the distribution of forecast error regression coefficients with simulated data from a FIRE model with regime shifts and then computing the significance level at which the empirical regression coefficient estimates allow one to reject the null of FIRE. The test is similar in spirit to simulation-based tests of rational expectations models with imperfect information and learning by Andolfatto et al. (2008) and Adam et al. (2017). Different from these tests, however, our test is applied to FIRE models with regime shifts, and takes into account not only finite sample uncertainty but also uncertainty about the data-generating process and uncertainty about the sequence of realized regimes.

Section 5 generalizes the analysis to any Markov-switching FIRE model with a minimum state variable solution. We show that ex-post forecast errors are typically a complicated function of the current state of the economy and the sequence of realized regimes over the entire forecast horizon. The result confirms the predictability of ex-post forecast errors in FIRE models with regime shifts. At the same time, the result implies that simple univariate forecast error regressions as used in literature are generally subject to omitted variable bias because the variables used in these regressions do not span the information set that agents use. This means that even if one abstracts from the fact that regime realizations are generally unobserved, forecast error regression estimates do not have a structural interpretation and are therefore not informative about the underlying expectations data-generating process.

Section 6, finally, assesses the extent to which a medium-scale New Keynesian model along the lines of Christiano et al. (2005), Smets and Wouters (2007), and Justiniano et al. (2011) augmented with Markov regime shifts in the monetary policy interest rate rule as proposed by Bianchi (2013) is quantitatively consistent with the empirical evidence on the predictability of forecast errors reviewed in Section 2. We estimate the model with Baysian likelihood-based techniques on U.S. macro aggregates and then apply the regime-robust test of FIRE. We find that based on this data-generating process, the test fails to reject the null of FIRE decisively. Conditional on the observed macro aggregates, the model also generates sizable waves of over- and under-reaction of expectations to current information over rolling sample windows. Regime shifts in monetary policy play only a small role for these waves, however, and the waves are generally quite different from the empirical estimates. This represents a clear challenge for the model, which is considered a benchmark for modern business cycle analysis and our understanding of monetary policy, thus providing empirical motivation to consider data-generating processes with a richer regime shift structure (e.g., in trend inflation and/or trend growth) and/or departures from FIRE.

The paper is related to several literatures. As reviewed in Section 2, the paper contributes to a burgeoning literature on the predictability of survey-based forecast errors of macro aggregates. The key insight of our analysis is that in the presence of regime shifts, predictable forecast errors is not a sufficient condition to reject FIRE. As already emphasized, we do not interpret this results as a critique of alternative theories of expectation formation. Our point instead is that reduced-form forecast error regressions by themselves are unlikely to inform the empirical validity of alternative theories relative to FIRE.

The result shares clear parallels with an earlier asset pricing literature on tests of the efficient markets hypothesis in the presence of so-called peso problems; i.e. anticipated changes in the probability distribution of asset prices. See for instance Rietz (1988); Engel and Hamilton (1990); Cecchetti et al. (1993); Kaminsky (1993); Evans and Lewis (1995a, 1995b); Bekaert et al. (2001);

or Barro (2006).[5] The main difference of our paper relative to this literature is that we study the consequences of regime shifts for the predictability of ex-post forecast errors in a modern Dynamic Stochastic General Equilibrium (DSGE) context, propose a formal regime-robust test of FIRE, and apply the test to an estimated medium-scale DSGE model with plausible regime shifts.

The paper also contributes to a recent literature that analyzes the extent to which learning in an equilibrium context can explain salient features of survey-based forecast errors of macroeconomic aggregates. Aside from work by Andolfatto et al. (2008) and Adam et al. (2017) mentioned above, the paper perhaps most closely related is King and Lu (2021) who propose a model with endogenous regime shifts in monetary policy and private sector learning to account for the rise and fall in U.S. inflation and the concomitant dynamics of inflation forecast errors in the SPF. Other related papers are Farmer et al. (2023) who propose a model of professional forecasters who learn about low-frequency features of the underlying data-generating process to account for various "forecast anomalies"; as well as Andolfatto and Gomme (2003); Davig (2004); Schorfheide (2005); Bullard and Singh (2012); Richter and Throckmorton (2015); and Foerster and Matthes (2022) among others who introduce imperfect information and learning into otherwise rational expectations DSGE models with Markov regime shifts. The distinguishing feature of our analysis is to show that even with perfectly informed rational agents, regime shifts can generate predictable forecast errors.

2 Empirical evidence on survey-based forecast errors

In this section, we provide a brief review of the empirical evidence on the predictability of survey-based forecast errors. Then we document that survey-based forecasts exhibit waves of over- and under-reaction to current information across rolling sample windows.

2.1 Reduced-form forecast error regressions

A large literature documents that survey-based expectations of macroeconomic aggregates are often biased and that ex-post forecast errors – the difference between actual realizations and

[5] The name peso problem goes back to the empirical puzzle that forward rates on the Mexican Peso traded below the dollar exchange rate for much of the early 1970s even though the Peso was pegged to the dollar. Then, in 1976, the Peso was allowed to float and depreciated by almost 50 percent. Ex-post, the forward-spot rate difference prior to the devaluation looks like a predictable forecast error, but ex-ante it is consistent with rational expectations under the assumption of regime shifts. See Lewis (2008) for a review.

ex-ante forecasts – are autocorrelated in systematic and quantitatively important ways. See for example the reviews by Croushore (2010) and Coibion et al. (2018) as well as the references therein. While these results were initially greeted with skepticism, they have over time gained increasing acceptance as evidence against FIRE, reflecting either inefficient use of information by forecasters (departures from rationality) or sticky information / costly information acquisition (departures from full information) or both.[6]

More recently, the literature has expanded on this empirical evidence by estimating linear regressions of ex-post forecast errors for prominent macroeconomic aggregates (e.g. inflation, output growth) on information available at the time of forecast. For instance, Angeletos et al. (2020) and Kohlhas and Walther (2021) among others estimate

$$y_{t+h} - F_t y_{t+h} = \theta + \gamma y_t + e_{t+h}, \tag{1}$$

where $y_{t+h} - F_t y_{t+h}$ denotes the ex-post forecast error about time $t+h$ realization of some macro aggregate of interest, y_{t+h}, relative to its forecast at the end of period t and beginning of period $(t+1)$, $F_t y_{t+h}$; y_t is the current realization known to agents at the time of forecast; and e_{t+h} is an error term. In turn, Coibion and Gorodnichenko (2015), followed by Bordalo et al. (2020), Angeletos et al. (2020), and Kohlhas and Walther (2021) among others, estimate

$$y_{t+h} - F_t y_{t+h} = \omega + \delta \left(F_t y_{t+h} - F_{t-1} y_{t+h} \right) + e_{t+h}, \tag{2}$$

where $F_t y_{t+h} - F_{t-1} y_{t+h}$ denotes the ex-ante forecast revisions reflecting news known to the agents at the time of forecast.[7]

The OLS estimate $\hat{\gamma}_T$ of regression (1) is often found to be negative, although the significance and even the sign of the estimate depends on the macro aggregate, forecast horizon, and sample period considered. The OLS estimate $\hat{\delta}_T$ of regression (2), by contrast, is typically positive and significant.[8] These estimates are frequently interpreted as evidence that agents simultaneously

[6]See Mincer and Zarnowitz (1969), Friedman (1980), Nordhaus (1987), Maddala (1991), Croushore (1998) or Schuh (2001) for early examples on the former perspective; and Mankiw and Reis (2002), Mankiw et al. (2003), Sims (2003), Woodford (2003), or Mackowiak and Wiederholt (2009) for early examples on the latter perspective.

[7]We note that $F_t y_{t+h}$ denotes the forecast about y_{t+h} given information available at the end of period $(t-1)$ and beginning of period t. Hence, the subscript t in F_t denotes the period when information becomes available to the professional forecasters (end of period t), and not the period when they report the forecast (beginning of period $t+1$). This notation is different from the one in Coibion and Gorodnichenko (2015), $F_t y_{t+3}$,where t denotes the period when forecasters report the forecast.

[8]Some studies compute forecast errors by averaging forecasts across survey participants, while other studies use

over-react to the current state of the economy but under-react to news, which has led different authors to propose new theories of expectations formation based on information rigidity (Angeletos et al. (2020)) or asymmetric attention (Kohlhas and Walther, 2021).

Table 1: Forecast error regression estimates for U.S. inflation and output growth

Panel A: $y_{t+4} - F_t y_{t+4} = \theta + \gamma y_t + e_{t+4}$

	Full sample 1970:2-2019:1			*Subsample 1983:1-2019:1*		
	$\hat{\gamma}_T$	$\sigma_{\hat{\gamma}_T}$	$p(\gamma = 0)$	$\hat{\gamma}_T$	$\sigma_{\hat{\gamma}_T}$	$p(\gamma = 0)$
Output growth	−0.105	0.065	0.107	−0.049	0.092	0.594
Inflation	0.049	0.070	0.480	−0.169	0.070	0.017

Panel B: $y_{t+4} - F_t y_{t+4} = \omega + \delta(F_t y_{t+4} - F_{t-1} y_{t+4}) + e_{t+4}$

	Full sample 1970:2-2019:1			*Subsample 1983:1-2019:1*		
	$\hat{\delta}_T$	$\sigma_{\hat{\delta}_T}$	$p(\delta = 0)$	$\hat{\delta}_T$	$\sigma_{\hat{\delta}_T}$	$p(\delta = 0)$
Output growth	0.717	0.232	0.002	0.507	0.299	0.092
Inflation	1.010	0.459	0.029	0.111	0.221	0.617

Notes: The table reports OLS coefficient estimates, HAC-robust standard errors, and p-values of the null that the coefficients are zero for regressions of four-quarter ahead ex-post forecast errors of U.S. inflation and U.S. output growth on current realizations and current forecast revisions of the two variables, respectively. See the text for details on the data construction. HAC-robust standard errors are computed using the Newey-West estimator with bandwith set equal to 5.

To fix ideas and set the stage for the rest of paper, we reproduce some of these regression estimates for inflation and output growth. Following Coibion and Gorodnichenko (2015), Angeletos et al. (2020), and Kohlhas and Walther (2021) among others, we use quarterly data from the SPF and focus on four-quarter ahead forecasts. The sample covers the period 1970:2-2019:1.[9] We measure inflation at time t (i.e. y_t in the above notation) as the average quarterly growth rate of the real-time GDP deflator over the last four quarters (i.e. time $t-4$ to t) and repeat the same computation with chain-weighted real GDP to measure output growth. To construct four-quarter ahead, that is annual, forecasts, we use the consensus forecasts and average the forecasts at time t about quarterly inflation (similarly for output growth rates) in the end of periods $(t+1)$, $(t+2)$,

individual forecasts and estimate the two regressions with individual fixed effects. The results are typically very similar. Bordalo et al. (2020) and others, in turn, estimate regression (2) as a panel using individual forecast errors $y_{t+h} - F_{it} y_{t+h}$ and individual forecast revisions $F_{it} y_{t+h} - F_{it-1} y_{t+h}$. They report negative as opposed to positive estimates of δ. As Angeletos et al. (2020) and Kohlhas and Walther (2021) point out, however, the sign of this estimate depends on the treatment of outliers in the individual forecast data and the sample period. We return to discussing evidence on individual forecasts and forecast dispersion towards the end of the paper.

[9]Note that to construct annual forecast error data, we use data realizations up to period 2020:1.

$(t{+}3)$, and $(t{+}4)$. We then compute forecast errors as the difference between the average quarterly growth rate of the real-time GDP deflator over the last four quarters (i.e. time $(t+1)$ to $(t+4)$) and the annual inflation forecast.[10] For all observed realizations, we use real-time data because final revised data may reflect reclassification and information not available at the time of forecast (see Croushore, 2010). Following the above literature, we do not correct the estimates for finite sample bias and use HAC-robust standard errors for inference.[11]

Table 1 reports the results, both for the full sample and what we call the post-1970s subsample that starts in 1983:1 and ends in 2019:1, a period that is associated with low inflation and low output growth volatility.

As shown in Panel A, while the OLS estimate $\hat{\gamma}_T$ of regression (1) is negative for both the full sample and the post-1970s sample for the case of output growth, the sign switches for the case of inflation. Except for the case of inflation for the post-1970 subsample, one cannot reject the null of zero prediction at high significance levels. As discussed in Kohlhas and Walther (2021), however, the negative sign and significance of $\hat{\gamma}_T$ is somewhat more robust for samples starting in the mid-1980s and ending before the 2008-09 Great Recession, and when inflation is measured with the consumer price index (CPI) as opposed to the GDP deflator.

As shown in Panel B, the OLS estimates $\hat{\delta}_T$ of regression (2) are generally positive and, at least for the full sample, highly significant, thus confirming the results in Coibion and Gorodnichenko (2015). At the same time, the magnitude of the estimates declines considerably for the post-1970s subsample and, for the case of inflation, the estimate becomes insignificant.

2.2 Waves of over- and under-reaction

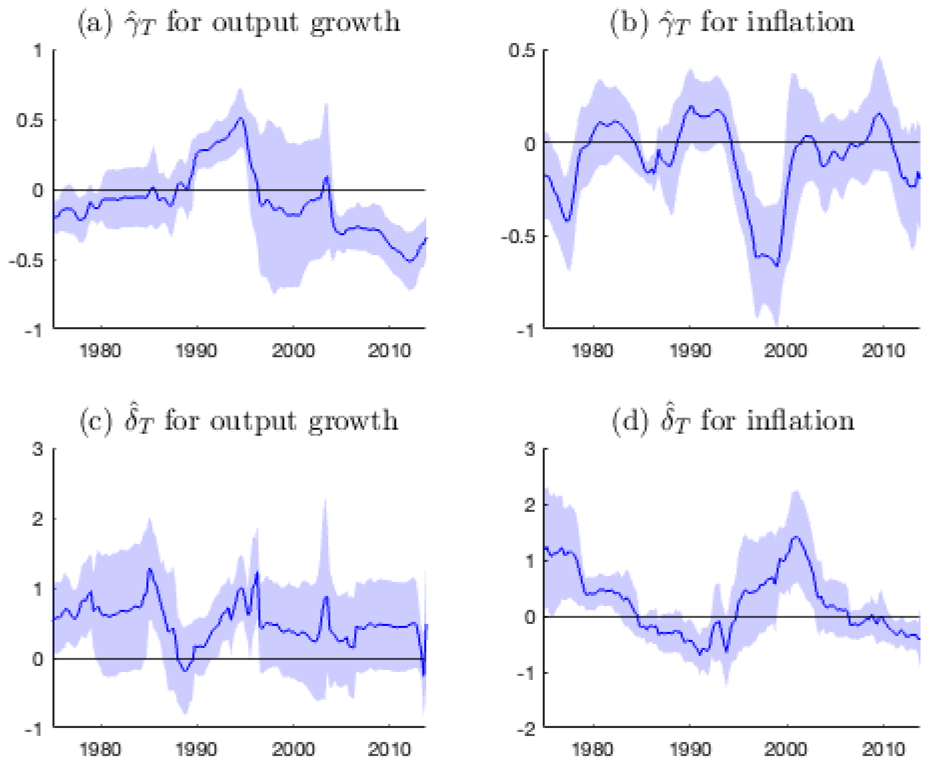
To investigate the variation in the predictability of forecast errors further, we estimate each of the above regressions over rolling 40-quarter samples. Figure 1 reports the point estimates (blue solid

[10]Similarly, to construct $F_{t-1}y_{t+4}$, we average the forecasts at time $(t-1)$ about quarterly inflation and output growth rates in the end of periods $(t+1)$, $(t+2)$, $(t+3)$, and $(t+4)$. The way we construct forecasts and forecast errors is similar to Coibion and Gorodnichenko (2015) and Kohlhas and Walther (2021), with some minor differences. The former compute annual forecast errors by averaging the one- through four-quarter ahead forecast errors. The latter, instead, rely on the forecast about the level of the GDP deflator and real output growth to construct forecasts about the annual growth rates, by computing the growth rate between the forecast about the level in period $(t+4)$ and the forecast about the level in period $(t+1)$.

[11]OLS coefficient estimates are biased in finite samples if the regressors are not strictly exogenous. As discussed in Section 3, strict exogeneity is typically violated for the type of regressions in (1) and (2), independent of whether the data-generating process contains regime shifts or not. Other studies such as Adam et al. (2017) bias-correct their estimates in a related context. The regime-shift robust test that we propose in Section 4 automatically takes finite sample bias into account.

lines) with associated 90% confidence intervals (blue shaded areas).

Figure 1: Waves of over- and under-reaction in SPF data



Notes: The plots show 40-quarter rolling regression coefficient estimates of four-quarter ahead ex-post forecast errors of U.S. output growth and U.S. inflation on current realizations and current forecast revisions of the two variables. See the text for details on the data construction. The blue shaded areas show 90% confidence bands based on HAC-robust standard errors computed using the Newey-West estimator with bandwidth set equal to 5. The estimates are centered at the midpoint of the rolling regression window (e.g. 1980 denotes the regression window 1975:1 to 1984:4).

The figure provides evidence of large waves of over- and under-reaction to current information. As shown in panels (a) and (c), forecast errors for output growth are essentially unrelated to current realizations from the 1970s to the early 1990s, positively associated during the 1990s, and and then negatively associated during the 2000s. In turn, forecast errors of output growth are mostly positively associated with current forecasts revisions, although there is a marked downward swing from the late 1980s through the mid-1990s and the estimates are generally surrounded by considerable uncertainty.

Panels (b) and (d) show even larger waves in the regression coefficients for inflation. During the 1970s and then again from the 1990s to the mid-2000s, inflation forecasts errors are predicted to be significantly negatively related to current realizations but significantly positively related to forecast revisions of inflation. In the 1980s as well as from the mid-2000s onward, inflation forecast errors are less strongly related to the two predictors.

We view these waves of over- and under-reaction across rolling sample windows as an interesting new stylized fact. On the one hand, some of the waves could be due to small sample uncertainty. On the other hand, the magnitude of the waves seems too large to be solely explained by data.[12] This represents a challenge for theories of forecast error predictability based on departures from FIRE alone as these theories imply constant over- or under-reaction to current information. In what follows, we therefore explore the potential of an alternative explanation based on regime shifts.[13]

3 Predictable forecast errors in a univariate model

This section considers a univariate FIRE model, first without regime shifts and then with regime shifts. While too simple from an empirical standpoint, the model has the advantage that the relationship of forecast errors with current information can be derived analytically and has clear intuition.

3.1 No regime shifts

Consider an endogenous variable of interest y_t with the following FIRE solution

$$y_t = ax_t, \tag{3}$$

where the exogenous variable x_t evolves according to

$$x_t = \phi x_{t-1} + \varepsilon_t, \tag{4}$$

[12]We formally explore this possibility in the next section. We also note that large waves of over- and under-reaction obtain with larger rolling windows (e.g., 60 quarters).

[13]We note that there are also large swings in the estimates of the regression constants $\hat{\theta}$ and $\hat{\omega}$ across rolling sample windows. This indicates time variation in the average bias of forecasts. Furthermore, the estimates $\hat{\gamma}$ and $\hat{\delta}$ can vary substantially depending on whether the regression includes additional variables (either other macro aggregates or lagged values of the variable forecasted). While less relevant for our motivation of exploring the implications of regime shifts, we show in the generalized framework of Section 5 how regime shifts in FIRE models can also lead to non-zero biases that are time-varying across rolling sample windows and coefficient instability with respect to additional regressors.

with $\phi \in [0,1)$ and $\varepsilon_t \sim i.i.d.(0,\sigma^2)$.[14]

Given (3) and (4), FIRE implies that for any horizon $h \geq 1$, agents' forecasts of x_{t+h} conditional on information at time t are

$$\mathbb{E}_t y_{t+h} = a\phi^h x_t, \tag{5}$$

and ex-post forecast errors can be expressed as

$$y_{t+h} - \mathbb{E}_t y_{t+h} = a\phi^h x_t + a\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau} - a\phi^h x_t = a\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau}. \tag{6}$$

In the absence of regime shifts, ex-post forecast errors under FIRE are a linear combination of i.i.d. innovations $\{\varepsilon_{t+\tau}\}_{\tau=1}^{h}$ that are unpredictable based on current information; i.e., $\mathbb{E}\left[\left(y_{t+h} - \mathbb{E}_t y_{t+h}\right) y_t\right] = a\mathbb{E}\left[\left(\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau}\right) y_t\right] = 0$. Intuitively, the portion of ex-post realization y_{t+h} that is predetermined as of t (i.e. $a\phi^h x_t$) is exactly the same as the agent's forecast based on information at t and thus, ex-post forecast errors are unpredictable. Similarly, forecast errors are equally unpredictable based on news as captured by ex-ante forecast revisions about y_{t+h} from time $t-1$ to time t,

$$\mathbb{E}_t y_{t+h} - \mathbb{E}_{t-1} y_{t+h} = a\phi^h \varepsilon_t. \tag{7}$$

As reviewed in Section 5, in the absence of regime shifts, the same result of unpredictable forecast errors holds for any linear FIRE model. The result constitutes the starting point for the above-discussed literature documenting that ex-post forecasting errors constructed from survey-data are in fact predictable, a finding that is typically taken as evidence against FIRE.

3.2 Markov regime shifts

Suppose instead that the FIRE solution in (3) switches between two regimes $s_t \in \{1,2\}$; i.e.

$$y_t = a_{s_t} x_t, \tag{8}$$

[14] Consider, for example, the expectational difference equation $y_t = \beta\mathbb{E}_t y_{t+1} + \psi x_t$ with $\mid \beta \mid < 1$, $\psi \geq 0$ and $\mathbb{E}_t$ denoting the rational expectations operator conditional on information at time t. Given the exogenous process in (4), the FIRE solution for this equation is (3) with $a = \frac{\psi}{1-\beta\phi}$.

where

$$a_{s_t} = \begin{cases} a_1 & \text{if } s_t = 1 \\ a_2 & \text{if } s_t = 2 \end{cases} \tag{9}$$

and that the regime switching is governed by an exogenous Markov process with transition matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \tag{10}$$

where $p_{ij} = Pr(s_t = j \mid s_{t-1} = i)$ with $0 < p_{ij} < 1$ and $\sum_{j=1}^{2} p_{ij} = 1$ for both $i, j = 1, 2$.[15] Two regimes are sufficient for the purpose of this illustration, though the results easily generalize to many regimes. Also, all results carry through if we allow for regime shifts in the persistence parameter ϕ of the exogenous process for x_t. In this section, we abstract from these generalizations to keep the example as simple as possible.

Given (8)-(10), FIRE implies that for any horizon $h \geq 1$, agents' forecasts of y_{t+h} conditional on information at time t (including regime realization s_t) are given by

$$\mathbb{E}_t y_{t+h} = \left(P^{(h)}_{s_t,1} a_1 + P^{(h)}_{s_t,2} a_2 \right) \phi^h x_t, \tag{11}$$

where $P^{(h)}_{s_t,s_{t+h}}$ is the (s_t, s_{t+h}) element of P^h. Hence, agents' expectations are a weighted average of regime-conditional forecasts: $a_1 \phi^h x_t$ if the first regime realizes in $t + h$, which occurs with probability $P^{(h)}_{s_t,1}$, and $a_2 \phi^h x_t$ if the second regime realizes in $t + h$, which occurs with probability $P^{(h)}_{s_t,2}$.

Based on (11), we derive the following key result:

PROPOSITION 1. *Given the exogenous forcing process (4) and regime-switching model described by (8)-(10), ex-post forecast errors under FIRE are related to current information by*

$$y_{t+h} - \mathbb{E}_t y_{t+h} = \gamma^{(h)}_{s_t,s_{t+h}} y_t + \xi_{t+h}, \tag{12}$$

where $\gamma^{(h)}_{s_t,s_{t+h}} \equiv \frac{(-1)^{s_{t+h}-1}(a_1-a_2)\left(1-P^{(h)}_{s_t,s_{t+h}}\right)\phi^h}{a_{s_t}}$*, and* $\xi_{t+h} \equiv a_{s_{t+h}} \sum_{\tau=1}^{h} \phi^{h-\tau} \varepsilon_{t+\tau}$ *is uncorrelated with*

[15] Returning to the example from the previous footnote, suppose the parameters $\{\beta, \psi\}$ of the expectational difference equation take on different values across the two regimes $s_t \in \{1, 2\}$. Then under conditions described in Davig and Leeper (2007), the FIRE solution takes the form in (9) with $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}' = (I_2 - \phi \boldsymbol{\beta} P)^{-1} \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix}'$ and $\boldsymbol{\beta} = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}$.

y_t. *Furthermore,* $\gamma^{(h)}_{s_t,s_{t+h}} = 0$ *for any* $h \geq 1$ *if and only if* $a_1 = a_2$ *or* $\phi = 0$.

Proof. See Appendix A.1. □

Proposition 1 establishes that in the presence of Markov regime shifts, ex-post forecasting errors are systematically predictable even though agents have full information and are fully rational. Intuitively, and in contrast to the case without regime shifts, the portion of y_{t+h} that is related to current information ($a_{s_{t+h}}\phi^h x_t$) differs from agents' forecast because, as described in (11), agents' expectations at time t are a weighted average of regime-conditional forecasts. Forecast errors, measured *ex-post* after regimes have realized, are therefore systematically related to information x_t available at the time of forecast.

Corollary 1 elaborates on the sign of $\gamma_{t,t+h}$.

Corollary 1. *Given the environment in Proposition 1,*

$$sign(\gamma^{(h)}_{s_t,s_{t+h}}) = \begin{cases} sign(a_1 - a_2) & \text{if } s_{t+h} = 1 \\ -sign(a_1 - a_2) & \text{if } s_{t+h} = 2 \end{cases} \tag{13}$$

Proof. See Appendix A.2. □

Without loss of generality, suppose from hereon that $a_1 > a_2$; i.e. the first regime is the one associated with a larger response to exogenous shocks. Hence, $\gamma^{(h)}_{s_t,s_{t+h}} > 0$ whenever $s_{t+h} = 1$. Absent regime shifts, a positive value of $\gamma^{(h)}_{s_t,s_{t+h}}$ would be interpreted as forecasters under-reacting to current information, either because of incomplete information or departures from rationality. According to Proposition 1 and Corollary 1, by contrast, this under-reaction occurs because fully informed rational agents ex-ante put non-zero probability on the less responsive regime, thus attenuating their forecast.

Similarly, we can derive the implications of regime shifts for the relation between ex-post forecast errors and news as captured by ex-ante forecast revisions $\mathbb{E}_t y_{t+h} - \mathbb{E}_{t-1} y_{t+h}$.

Proposition 2. *Given the same environment as in Proposition 1, ex-post forecast errors under FIRE are related to ex-ante forecast revisions by*

$$y_{t+h} - \mathbb{E}_t y_{t+h} = \delta^{(h)}_{s_t,s_{t+h}} \left(\mathbb{E}_t y_{t+h} - \mathbb{E}_{t-1} y_{t+h}\right) + \lambda^{(h+1)}_{s_{t-1},s_{t+h}} y_{t-1} + \xi_{t+h}, \tag{14}$$

where $\delta^{(h)}_{s_t,s_{t+h}} \equiv \frac{(-1)^{s_{t+h}-1}(a_1-a_2)\left(1-P^{(h)}_{s_t,s_{t+h}}\right)}{P^{(h)}_{s_t:}\boldsymbol{a}}$, $\lambda^{(h+1)}_{s_{t-1},s_{t+h}} \equiv \frac{(-1)^{s_{t+h}-1}(a_1-a_2)\left(1-P^{(h)}_{s_t,s_{t+h}}\right)P^{(h+1)}_{s_{t-1}:}\boldsymbol{a}}{\phi a_{s_{t-1}}P^{(h+1)}_{s_{t-1}:}\boldsymbol{a}}$, *and* ξ_{t+h} *is defined as in Proposition 1.*

Proof. See Appendix A.3. □

Proposition 2 establishes that in the presence of Markov regime shifts, ex-post forecast errors under FIRE are also systematically predictable by ex-ante forecast revisions as well as lagged information. Moreover, note that for this simple univariate model, the sign of $\delta^{(h)}_{s_t,s_{t+h}}$ is the same as the sign of $\gamma^{(h)}_{s_t,s_{t+h}}$ given in Corollary 1. As we shall see in Section 5, this result does not necessarily hold for richer FIRE models with regime shifts.

3.3 Implications for reduced-form forecast error regressions

Given Propositions 1 and 2, we now study the implications of regime shifts for the type of reduced-form forecast error regressions reported in the literature. Consider first estimating a demeaned version of regression (1),

$$y_{t+h} - F_t y_{t+h} = \gamma y_t + e_{t+h}, \tag{15}$$

from sample $\{y_t, y_{t+h} - F_t y_{t+h}\}_{t=1}^{T}$ generated by the regime-switching model in (8)-(10) *conditional on* sequence of regime realizations $\{s_t\}_{t=1}^{T+h}$.[16] Under the assumption that forecasters are fully informed rational expectations agents (i.e. $F_t = \mathbb{E}_t$), the expected ordinary least square (OLS) estimate of γ can be approximated by

$$\mathbb{E}\left[\hat{\gamma}_T | \{s_t\}_{t=1}^{T+h}\right] \approx \frac{\sum_{i=1}^{2}\sum_{j=1}^{2} a_i^2 \gamma_{ij}^{(h)} \mathcal{F}_T^{(h)}(i,j)}{\sum_{i=1}^{2} a_i^2 \mathcal{F}_T(i)} + fsb_T, \tag{16}$$

where a_i is defined as in (9); $\gamma_{ij}^{(h)}$ describes the relation between $y_{t+h} - \mathbb{E}_t y_{t+h}$ and y_t conditional on regime realizations $s_t = i$ and $s_{t+h} = j$ as defined in Proposition 1; $\mathcal{F}_T^{(h)}(i,j) \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}(s_t = i, s_{t+h} = j)$ is the sample frequency of these joint regime realizations occurring; $\mathcal{F}_T(i) \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}(s_t = i)$ is the unconditional sample frequency of regime realizations $s_t = i$; and fsb_T denotes the expected finite sample bias due to the fact that the regressor y_t is not strictly

[16]Using demeaned data is consistent with the FIRE solution of the data-generating process in (8). This type of equation generally arises as the result of log-linearizing optimality conditions of dynamic stochastic problems, which by definition refer to deviations from the mean. Section 5 considers the a generalized framework that includes regime shifts in constant terms.

exogenous.[17] Appendix A.4 provides details of the derivation. The approximation stems from the fact that both $\gamma_{ij}^{(h)}$ and y_t are a function of regime realizations and are therefore not independent of each other. However, for sufficiently persistent regime processes as estimated in the application below, (16) provides a very good characterization of $\mathbb{E}\left[\hat{\gamma}_T | \{s_t\}_{t=1}^{T+h}\right]$.

Note that the sample frequency of joint regime realizations can be expressed as $\mathcal{F}_T^{(h)}(i,j) = f_{ij}^{(h)}\mathcal{F}_T^{(h)}(j)$, where $f_{ij}^{(h)} \equiv \frac{1}{T}\sum_{t=1}^{T}\mathbf{1}(s_t = i | s_{t+h} = j)$ is the sample frequency of regime realization $s_t = i$ conditional on regime realization $s_{t+h} = j$. As shown in Appendix A.4, we can therefore rewrite the first part of (16) as

$$\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] = \underbrace{\frac{\phi^h(a_1 - a_2)}{a_1^2(1 - f_{22}^{(h)}) + a_2^2(1 - f_{11}^{(h)})}}_{(+)} \underbrace{\left[a_1(1 - f_{22}^{(h)})\left(f_{11}^{(h)} - p_{11}^{(h)}\right) - a_2(1 - f_{11}^{(h)})\left(f_{22}^{(h)} - p_{22}^{(h)}\right)\right]}_{g(f_{11}^{(h)}, f_{22}^{(h)})}, \tag{17}$$

where $\hat{\gamma}_T^c$ denotes the fact that this is the bias-corrected OLS estimate. By assumption of $a_1 > a_2$, the first part of this expression is positive. Hence, the sign of $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right]$ is determined by the sign of $g(f_{11}^{(h)}, f_{22}^{(h)})$. This gives rise to the following proposition:

PROPOSITION 3. *Consider the same conditions as in Proposition 1 and assume without loss of generality that $a_1 > a_2$. Then under the null hypothesis of FIRE,*

1. *for a finite sequence of $\{s_t\}_{t=1}^{T+h}$ characterized by conditional sample frequencies $f_{11}^{(h)}$ and $f_{22}^{(h)}$,*

$$\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] \gtreqless 0 \Leftrightarrow f_{11}^{(h)} \gtreqless g(f_{22}^{(h)}) \equiv \frac{a_1 p_{11}^{(h)}(1 - f_{22}^{(h)}) + a_2(f_{22}^{(h)} - p_{22}^{(h)})}{a_1(1 - f_{22}^{(h)}) + a_2(f_{22}^{(h)} - p_{22}^{(h)})};$$

2. *for $T \to \infty$, $g(f_{11}^{(h)}, f_{22}^{(h)}) \to 0$ and $fsb_T \to 0 \Rightarrow \mathbb{E}\left[\hat{\gamma}_T | \{s_t\}_{t=1}^{T+h}\right] \to \mathbb{E}[\gamma] = 0$.*

Proof. See Appendix A.4. □

The first part of the proposition establishes that in the presence of regime shifts, unpredictability of forecast errors is a knife-edge case. Generally, regime realizations $\{s_t\}_{t=1}^{T+h}$ are such that either $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] < 0$ or $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] > 0$; i.e. agents look like they over- or under-react to current information y_t. Intuitively, $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right]$ is a weighted average of the four possible values

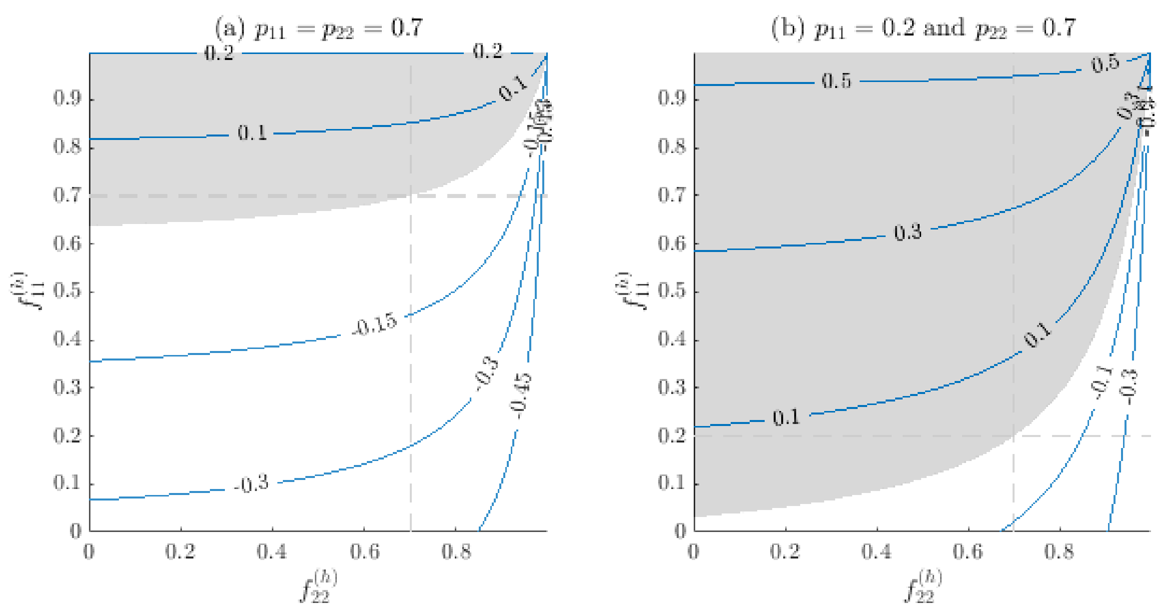
[17] By definition, we have $\mathbb{E}(y_t e_{t+h}) = 0$; but due to the autoregressive nature of the forcing process (4), $\mathbb{E}(y_{t+k}e_{t+h}) \neq 0$ for any $k > 0$. Hence, the OLS assumption for unbiasedness is violated in finite samples. A similar bias is also present in the absence of regime switching. As discussed in Section 2, the empirical literature often ignores this bias.

of γ_{ij}, two of which are positive (γ_{11} and γ_{21}) and two of which are negative (γ_{12} and γ_{22}) . The weights and therefore the sign of $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right]$ depend on the conditional sample frequencies $f_{11}^{(h)}$ and $f_{22}^{(h)}$ of regime realizations $\{s_t\}_{t=1}^{T+h}$ relative to what agents expect ($p_{11}^{(h)}$ and $p_{22}^{(h)}$) as well as on a_1 and a_2.

The second part of the proposition establishes that as T increases, the conditional sample frequencies of regime realizations converge to their population counterparts and therefore agents' expectations. Hence, periods of over- and under-reaction tend to cancel each other out such that in the limit, ex-post forecast errors become unpredictable. Similarly, the finite sample bias vanishes in the limit. The result makes clear that ex-post forecast error predictability remains a finite sample phenomenon, thus providing a potential new explanation for the observation that in survey expectation data, ex-post forecast error predictability often declines with longer time series (e.g. Croushore, 1998).

To explore Proposition 3 further, we set the univariate model parameters to $a_1 = 2$, $a_2 = 0.5$, $\phi = 0.9$ and consider two sets of transition probabilities: (a) $p_{11} = p_{22} = 0.7$; and (b) $p_{11} = 0.2, p_{22} = 0.7$. For each case, we compute $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right]$ for forecast horizon $h = 1$ across conditional sample frequencies $f_{11}^{(h)}$ and $f_{22}^{(h)}$. Figure 2 visualizes the result.

Figure 2: Apparent over- and under-reaction in the presence of regime shifts



Notes: The plots in two panels show the sign of expected bias-corrected OLS coefficients of regression (15) across conditional regime realizations $f_{11}^{(h)}$ and $f_{22}^{(h)}$ for different regime transition probabilities p_{11} and p_{22}. In each plot, the grey region shows combinations for which $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] > 0$; and the white region shows combinations for which $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] < 0$. The two regions are separated by the hyperbola $f_{11}^{(h)} = g(f_{22}^{(h)})$ in Proposition 3.

The knife-edge case for which $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] = 0$ is given by the hyperbola $f_{11}^{(h)} = g(f_{22}^{(h)})$ separating the grey from the white region. In the grey region, $f_{11}^{(h)} > g(f_{22}^{(h)})$ and thus $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] > 0$ (apparent under-reaction to current information); while in the white region, $f_{11}^{(h)} < g(f_{22}^{(h)})$ and thus $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] < 0$ (apparent over-reaction to current information). As the different contours show, the magnitude of these deviations from zero can be substantial even for relatively small differences of $f_{11}^{(h)}$ from $p_{11}^{(h)}$, respectively $f_{22}^{(h)}$ from $p_{22}^{(h)}$.

Proposition 3 implies that, as $f_{11}^{(h)}$ and $f_{22}^{(h)}$ vary across samples, regime shifts can lead to waves of over- and under-reaction. To illustrate this point, we simulate the univariate model for 500 periods, first without regime shifts and then with regime shifts. [18]We then estimate (15) for rolling window samples of $T = 40$ periods and, for each of the samples, correct the OLS point estimate for finite sample bias.[19] To compute the coverage bands, we perform a blind boostrapping procedure in order to preserve the regime path pertaining to each rolling sample. [20]

Figure 3 reports the results. As shown in panel (a), in the absence of regime shifts, the bias-corrected OLS point estimates are almost never significantly different from zero. This confirms that in the absence of regime shifts, ex-post predictability of forecast errors is a sufficient condition to reject FIRE even in relatively small samples.[21]

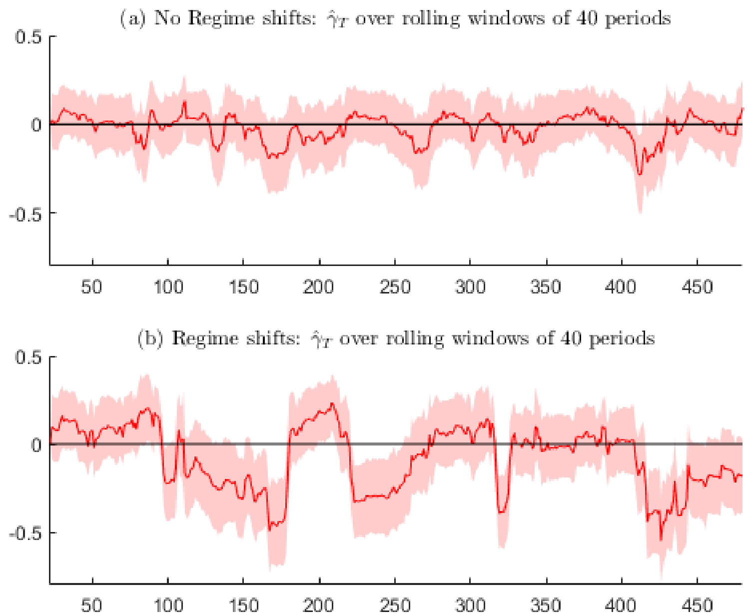
[18]For the simulation without regime shifts, we set $a = 1.25$ and choose $\phi = 0.9$, $\sigma_\varepsilon = 1$. For the simulation with regime shifts, we keep the exogenous forcing process unchanged and set $a_1 = 2$, $a_2 = 0.5$ with Markov transition probabilities $p_{11} = p_{22} = 0.7$. Similar results would obtain for other parameterizations.

[19]To correct for finite sample bias, we simulate $i = 1, ..., 10,000$ new samples of 500 periods, preserving the original sequence of regime realizations for each of the samples. For each 40-period rolling window, we then compute the model-implied finite sample bias $fsb_T^i = \frac{\sum_{\tau=t}^{t+39}(y_\tau^i - \bar{y}_{t:t+39}^i)(\xi_{\tau+1}^i - \bar{\xi}_{t+1:t+40}^i)}{\sum_{\tau=t}^{t+39}(y_{\tau,n} - \bar{y}_{t:t+39,n})^2}$ across the different samples i, and subtract the average bias $fsb_T = \sum_{i=1}^{10,000} fsb_T^i / 10,000$ from the OLS estimate.

[20]Using the bias-corrected OLS point estimates, we compute the fitted values of the regression in 15 as well as the standard deviation of the regression error terms for each rolling sample of simulated data. From a normal distribution with that standard deviation and mean 0, we generate $N = 1000$ i.d.d. innovations, and add those disturbances to the fitted values to generate 1000 new datasets of the dependent variable. Preserving the regressor, we re-estimate 15 and bias-correct the point estimates for finite sample bias using the model-implied bias averaged across the 1000 simulations. Finally, for each rolling sample we isolate the bottom and top 5% bias-free estimates to compute the 90% coverage bands.

[21][CHECK] Finite sample bias correction is important for this result. For the present simulation, the mean absolute bias across all 40-period rolling windows is −0.1. Hence, without bias-correction, a researcher would frequently reject zero ex-post predictability at standard significance levels even though this is the implication of the data-generating process.

Figure 3: Waves of over- and under-reaction in simulated data



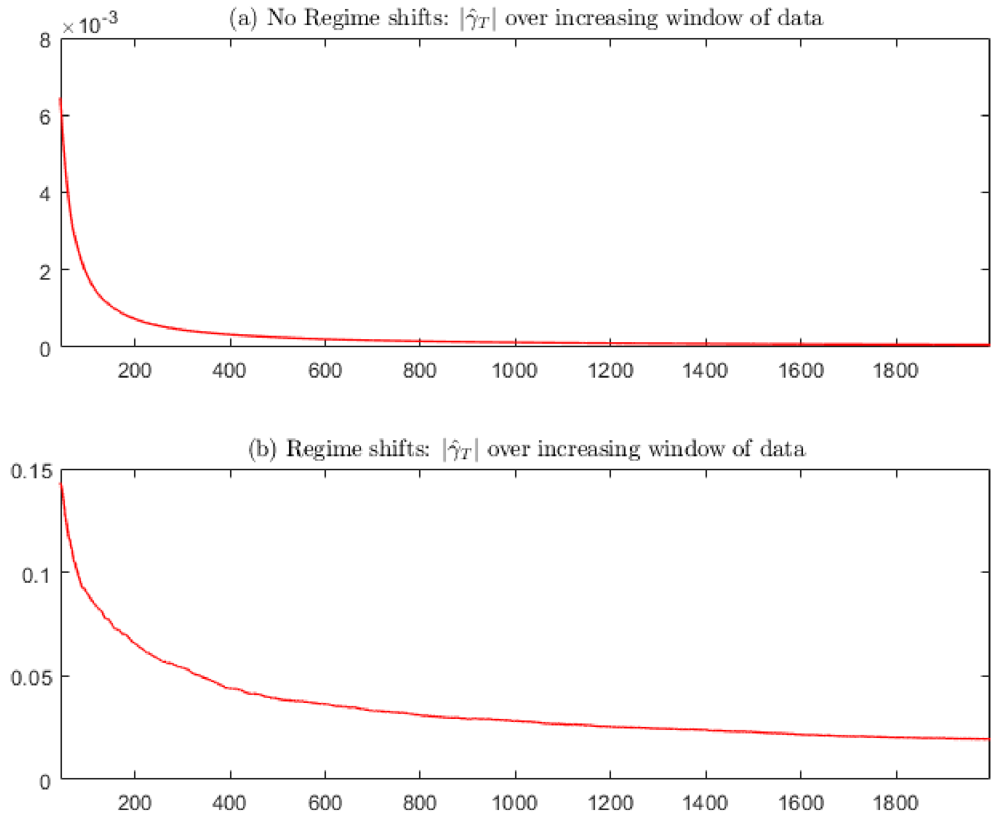
Notes: Panel (a) shows average bias-corrected OLS coefficient estimates and 90% coverage bands of regression (15) for rolling windows of 40 periods with data generated from the univariate model under FIRE without regime shifts. Panel (b) shows resuls for the same rolling window regressions, but with data generated from the univariate model under FIRE with regime shifts.

Panel (b) shows bias-corrected estimates for the data generated with regime shifts. There are much larger and often significant swings across the rolling sample windows.[22] This illustrates that regime shifts can lead to waves of over- and under-reaction across rolling sample windows and thus, ex-post predictability of forecast errors is not a sufficient condition to reject FIRE.

Proposition 3 also raises the question of what sample size T is large enough for ex-post forecast error predictability to vanish. To provide an answer, we simulate the univariate model 10,000 times for different sample sizes, each time with a different sequence of regime realization drawn from transition matrix P. For each sample size T, we then average the absolute values of the bias-corrected OLS estimates $\hat{\gamma}_T$ across simulations.

[22] [PROVIDE EXPLANATION] The swings are, on average, negative because for the particular calibrations chosen, $f_{11}^{(h)} > g(f_{22}^{(h)})$ and therefore $\hat{\gamma}_T < 0$ on average.

Figure 4: Average predictability of ex-post forecast errors by sample size



Notes: Panel (a) shows average absolute values of bias-corrected OLS estimates $\hat{\gamma}_T$ for different sample sizes T with data generated from the univariate model under FIRE without regime shifts. Panel (b) shows results for the same regressions, but with data generated from the univariate model under FIRE with regime shifts.

As shown in Figure 4, deviations from the asymptotic value of $\gamma = 0$ are on average small for the no-regime shift case, even in small samples, and $\hat{\gamma}_T$ converges quickly to 0. For the case of regime shifts, in contrast, deviations from the asymptotic value of $\gamma = 0$ are an order of magnitude larger and die out at a lower rate as T becomes large. Hence, at least in the context of the univariate model, the typical sample size for which we have expectations data of macroeconomic aggregates (100 to 200 quarters) is unlikely to be large enough for reduced-form forecast error regressions to be characterized by asymptotic properties.

Finally, consider estimating a demeaned version of regression (2),

$$y_{t+h} - F_t y_{t+h} = \delta \left(F_t y_{t+h} - F_{t-1} y_{t+h} \right) + e_{t+h}. \tag{18}$$

Different from regression (15), there is no equivalent closed-form solution that allows one to express the expected coefficient estimate $\mathbb{E}\left[\hat{\delta}_T | \{s_t\}_{t=1}^{T+h}\right]$ as a weighted average of the four values of $\delta_{ij}^{(h)}$ describing the relation between forecast errors $y_{t+h} - \mathbb{E}_t y_{t+h}$ and forecast revisions $\mathbb{E}_t y_{t+h} - \mathbb{E}_{t-1} y_{t+h}$ conditional on regime realizations $s_t = i$ and $s_{t+h} = j$. This is because, according to Proposition 2, forecast errors are a function of not only forecast revisions but also lagged outcomes y_{t-1}, and because forecast revisions by themselves do not span the information set that agents use to forecast y_{t+h}.[23] Hence, the coefficient estimate of δ is subject to omitted variable bias. As we shall see in Section 5, for more general, multivariate FIRE models with regime shifts, this omitted variable bias issue is even more pervasive and applies not only to forecast error regressions on forecast revisions as in (15) but also on forecast error regressions on current realizations as in (18).

Summing up, the analysis of this section yields two basic insights. First, in the presence of regime shifts, forecast error predictability is not a sufficient condition to reject FIRE. Second, in the presence of regime shifts, the coefficient estimates of the type of forecast error regressions used in the literature are complicated functions of the sample sequence of regime realizations and are generally subject to omitted variable bias. While we derive this result for the case of FIRE, the same result would obtain under the assumption of imperfect information or departures from rationality as long as expectations are at least partly forward-looking. This implies that forecast error regressions by themselves are not informative about alternative theories of expectations formation. We view this as an important point since the recent empirical literature has used these regressions to argue in favor or against specific forms of information frictions and departures from rationality (e.g., Coibion and Gorodnichenko, 2015; Angeletos et al., 2020; or Kohlhas and Walther, 2021).

4 A regime-shift robust test of FIRE

The above results imply that in the presence of regime shifts, standard statistical tests of FIRE based on reduced-form regressions are misspecified, both because the null of unpredictability of forecast errors is violated in finite samples, and because the usual standard errors do not take into account the uncertainty implied by regime shifts. Here, we propose a new regime-robust test that – given an underlying data-generating process (DGP) – allows one to assess the FIRE hypothesis with the type of reduced-form regressions used in the literature. The test is similar in spirit to

[23] To see this second point, note that by (11), $\mathbb{E}_t y_{t+h} - \mathbb{E}_{t-1} y_{t+h}$ is a function of not only x_t but also x_{t-1}. Hence, information of $\mathbb{E}_t y_{t+h} - \mathbb{E}_{t-1} y_{t+h}$ alone does not allow the econometrician to predict forecast errors.

simulation-based tests of RE models with imperfect information by Andolfatto et al. (2008) or Adam et al. (2017). Different from these tests, however, our test is applied to FIRE models with regime shifts and takes into account not only finite sample uncertainty but also uncertainty about the DGP and uncertainty about the regime path. Here, we illustrate the test with the simple univariate model from the previous section. In Section 6, we then apply the test to the empirically more relevant case of a medium-scale DSGE model.

Consider the univariate FIRE model with regime shifts given by (4) and (8)-(10), with uncertainty about this DGP characterized by posterior parameter distribution $P(\Theta|\mathcal{Z})$ estimated based on data $\mathcal{Z}$.[24] To simulate the finite sample distribution of the reduced form regression estimates $\hat{\gamma}_T$ and $\hat{\delta}_T$ under this null, we proceed in three steps: (i) draw $n = 1, ..., N$ parameter vectors Θ^n from $P(\Theta|\mathcal{Z})$; (ii) for each Θ^n, simulate $k = 1, ..., K$ samples of observations $\left\{y_t^{n,k}\right\}_{t=1}^{T+h}$ and $\left\{\mathbb{E}_t y_{t+h}^{n,k}\right\}_{t=1}^{T}$ from the DGP defined by Θ^n; and (iii) estimate $\hat{\gamma}_T^{n,k}$ and $\hat{\delta}_T^{n,k}$ for each of these samples. The resulting distributions can then be used to compute the probability that the simulated $\hat{\gamma}_T^{n,k}$, respectively $\hat{\delta}_T^{n,k}$, are larger in absolute value than the $\hat{\gamma}_T$, respectively $\hat{\delta}_T$, estimated from observed data. This provides a $p - value$ for a t-test of the null of FIRE.

Several comments are in order about this procedure. First, by simulating artificial samples based on different parameter draws of $P(\Theta|\mathcal{Z})$, the procedure incorporates uncertainty about the DGP. Second, the simulation of artificial samples in step (ii) is non-standard because of the need to incorporate the uncertainty about the sequence of realized regimes. We do so by drawing regime realizations $\left\{s_t^{n,k}\right\}_{t=1}^{T+h}$ for each sample k from the smoothed probabilities $\hat{Pr}(s_t \mid \mathcal{Z}_T; \Theta^n)$ implied by the DGP and the data $\mathcal{Z}_T$ over the sample period $t = 1, ...T$ used to estimate the reduced-form regressions.[25] Third, the procedure naturally provides us with the finite sample distribution of regression coefficients $\hat{\gamma}_T$ and $\hat{\delta}_T$. Hence, we do not need to bias-correct the estimates (e.g., through bootstrapping), and we can conduct inference without assuming normality.

To implement the procedure, we estimate $P(\Theta|\mathcal{Z})$ with standard Bayesian techniques using data for U.S. output growth from 1969:3 to 2020:1. The posterior modes of the different parameters are: $\phi = 0.87$, $\sigma = 0.29$, $a_1 = 5.27$, $a_2 = 1.77$, $p_{11} = 0.98$, and $p_{22} = 0.98$; i.e. the estimation

[24] For the univariate FIRE model with regime shifts, $\Theta = [\phi, \sigma, a_1, a_2, p_{11}, p_{22}]'$. Given that under the null, there exists an invariant DGP, the data $\mathcal{Z}$ used to estimate $P(\Theta|\mathcal{Z})$ may cover a larger period than the sample $\mathcal{Y}_T$ used to estimate $\hat{\gamma}_T$ and $\hat{\delta}_T$. This data may also include different variables than the ones used to estimate $\hat{\gamma}_T$ and $\hat{\delta}_T$.

[25] Alternatively, for each parameter draw Θ^n and simulation run k, one could draw unconditional regime realizations using the Markov transition matrix associated with Θ^n. However, this approach would not take into account that the test we want to implement is conditional on a sequence of regime realizations associated with the sample period $t = 1, ...T$.

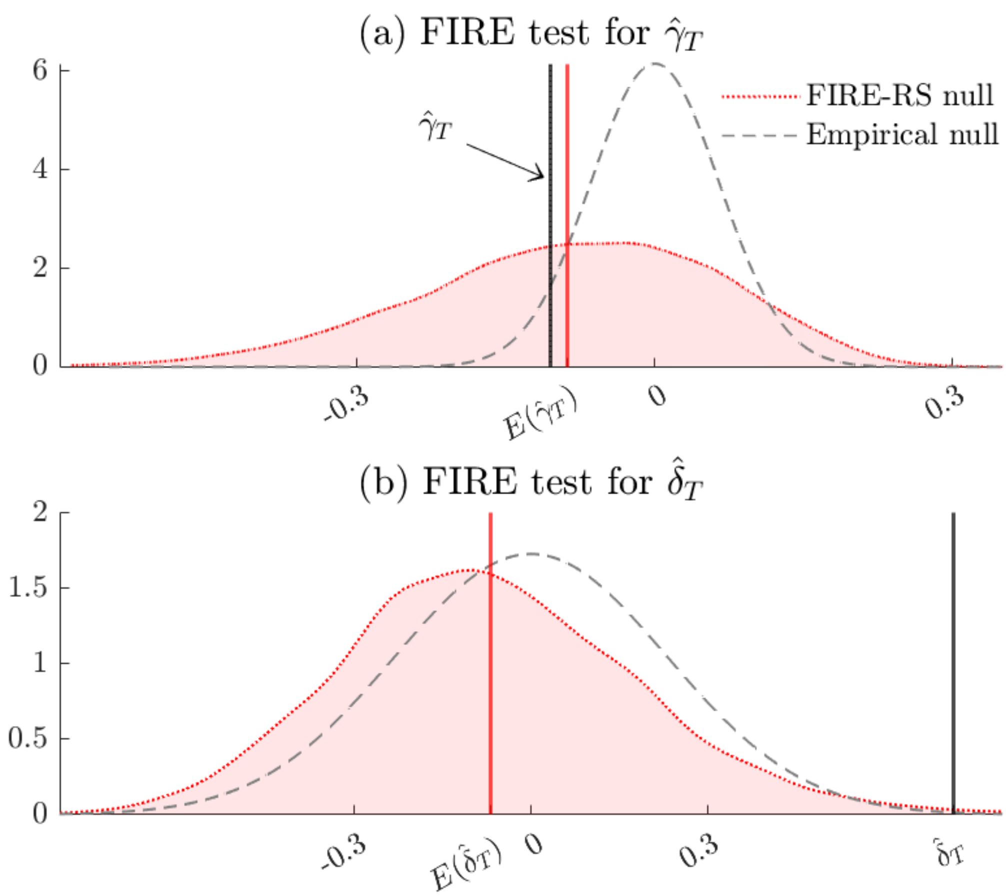
attributes very strong persistence to both regimes.[26] Conditional on this DGP, we then compute the test over the sample 1970:2-2019:1 used in Section 2 to estimate the forecast error regressions, with forecast horizon set to $h = 4$.

Figure 5 visualizes the results. The solid black lines show the OLS point estimates $\hat{\gamma}_T = -0.105$, respectively $\hat{\delta}_T = 0.717$ from Table 1, and the dashed black lines show the HAC-robust distributions of these estimates under the standard null of $H_0 : \gamma = 0$ and $H_0 : \delta = 0$. The red shaded areas show the simulated distributions of $\hat{\gamma}_T$ and $\hat{\delta}_T$ under the null that the data was generated by the estimated univariate FIRE model with regime shifts. These distributions are shifted to the left of the empirical distributions with $\mathbb{E}[\hat{\gamma}_T] < 0$ and $\mathbb{E}\left[\hat{\delta}_T\right] < 0$ both because of negative finite sample bias and because $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] < 0$ on average across the simulated sequences of $\{s_t\}_{t=1}^{T+h}$.[27] This is not a general result, however. As discussed above and elaborated upon further in the next section, $\mathbb{E}[\hat{\gamma}_T]$ and $\mathbb{E}\left[\hat{\delta}_T\right]$ can differ in sign due to omitted variable bias, depending on the DGP and the distribution of regime sequences $\{s_t\}_{t=1}^{T+h}$ across simulations.

[26] The estimation can be implemented either by treating x_t as unobserved or by measuring x_t with an observable. Since we do not want to impose additional assumptions on the nature of x_t, we treat it as unobserved and use the Kalman and Hamilton Filters as described in Kim and Nelson (1999) for estimation. See Appendix B for details. Note that the data strongly prefers the univariate model with regime shifts over the alternative without regime shifts.

[27] There are two reasons why $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] < 0$ on average in the present case. First, the conditional sample frequencies $f_{11}^{(h)}$ and $f_{22}^{(h)}$ implied by the simulated sequences of $\{s_t\}_{t=1}^{T+h}$ are on average smaller than the estimated transition probabilities $p_{11}^{(h)}$ and $p_{22}^{(h)}$ that agents use according to the model to form expectations. Second, $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right]$ is a non-linear function of $f_{11}^{(h)}$, $f_{22}^{(h)}$, $p_{11}^{(h)}$ and $p_{22}^{(h)}$. Hence, even if $f_{11}^{(h)}$ equaled $p_{11}^{(h)}$ and $f_{22}^{(h)}$ equaled $p_{22}^{(h)}$ on average, $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right]$ would on average not equal zero. Appendix B provides further analysis of this result.

Figure 5: Regime-robust test of FIRE for the case of output growth



Notes: Panel (a) shows the sample distributions of the OLS estimate of forecast error regression (1) for the case of output growth under the usual empirical null of $\mathbb{E}\left[\hat{\gamma}_T\right] = 0$ and HAC-robust standard errors (dashed black lines) and under the null that the data was generated by the univariate FIRE model with regime shifts (shaded red area). Panel (b) shows the corresponding distributions of the OLS estimate of forecast error regression (2). The sample for both panels (a) and (b) is 1970:2-2019:1.

As shown in panel (a), the OLS estimate of $\hat{\gamma}_T$ is in the left tail of the standard distribution associated with $H_0 : \gamma = 0$, implying a p-value of 0.11 in a two-sided t-test (twice the area under the dashed distribution to the left of $\hat{\gamma}_T$). According to the usual assumption of unpredictable forecast errors (and ignoring finite sample bias), a researcher would therefore reject the null of FIRE at a significance level of 11%.

Based on the regime-robust test with the univariate FIRE model, in contrast, the estimate of $\hat{\gamma}_T$ implies a p-value of 0.45 (the area under the shaded distribution to the left of $\hat{\gamma}_T$ plus the corresponding area to the right of $2\left(\mathbb{E}\left[\hat{\gamma}_T\right] - \hat{\gamma}_T\right)$. Hence, a researcher would not be able to reject the null of FIRE at a reasonable significance level. There are two reasons for this difference. First, the distribution is shifted to the left of the standard null; and second, the distribution is wider than what is implied by HAC-robust standard errors.

Turning to panel (b), the OLS estimate of $\hat{\delta}_T$ is in the far right tail of not only the standard HAC-robust distribution associated with $H_0 : \delta = 0$ but also the distribution implied by the estimated univariate FIRE model with regime shifts. Hence, the p-value associated with both the empirical distribution and the simulated distribution is essentially 0 under either null. Based on the regression of forecast errors on forecast revisions, a researcher would therefore reject FIRE with a high degree of confidence.

It is important to emphasize, however, that this rejection is conditional on the particular univariate DGP, estimated using U.S. output growth data for 1969:4-2020:1; i.e., a sample that is essentially the same as the sample over which we simulate data for the reduced-form regressions. As a result, the simulated conditional frequencies $f_{11}^{(h)}$ and $f_{22}^{(h)}$ are on average close to the $p_{11}^{(h)}$ and $p_{22}^{(h)}$ that agents use to form expectations, implying that regime switching imparts relatively little departure of $\mathbb{E}\left[\hat{\gamma}_T\right]$ and $\mathbb{E}\left[\hat{\delta}_T\right]$ from zero. Moreover, as documented extensively in the literature (e.g., Stock and Watson, 2003), U.S. output growth during the period experienced essentially two phases: a high-volatility phase that lasted up to the early 1980s and a low-volatility phase, commonly known as the Great Moderation, that extended from the early-1980s up to the 2008-09 financial crisis. As a result, the two regimes are not only estimated to be very persistent, as evidenced by the posterior modes $p_{11} = 0.98$, and $p_{22} = 0.98$, but there is also very little uncertainty surrounding these estimates (see Appendix B). The sequences of realized regimes $\left\{s_t^{n,k}\right\}_{t=1}^{T+h}$ implied by $\hat{Pr}(s_t \mid \mathcal{Z}_T; \Theta^n)$ are therefore close to invariant across simulations, and the distributions of $f_{11}^{(h)}$ and $f_{22}^{(h)}$ are narrow. Hence, regime switching in this particular application also imparts relatively little uncertainty about $\hat{\gamma}_T$ and $\hat{\delta}_T$ under the null.

Yet, as suggested by U.S. experience since the 2008-09 financial crisis as well as the experience of other countries, it is conceivable that these estimates of the regime switching process may not be reflective of the true regime switching process and therefore agents' expectations under FIRE. An econometrician may therefore want to consider a wider distribution of p_{11} and p_{22} that is, in addition, centered at lower values. When doing so, we find that it is relatively easy to end up with a distribution for $\hat{\gamma}_T$ and $\hat{\delta}_T$ under the null that is both substantially wider and shifted to the right so that it is no longer possible to reject the hypothesis of FIRE for neither of the two regressions. We return to this point in Section 6 when we implement the proposed test of FIRE with an estimated medium-scale DSGE model.

5 Generalized Framework

The above results are useful for understanding the nature of forecast error predictability implied by regime shifts in a simple univariate context. In this section, we show that ex-post predictability of forecast errors is a generic feature of the FIRE hypothesis in any model with regime shifts that has a minimum state variable (MSV) solution. Importantly, we also show that forecast errors can no longer be represented by a univariate equation. Instead, the complexity of the ex-post forecast errors representation increases with that of the underlying DGP, which has several important implications.

5.1 Environment

The MSV solution to any FIRE model with regime shifts can be expressed as

$$X_t = C_{s_t} + A_{s_t} X_{t-1} + B_{s_t} \epsilon_t \tag{19}$$

where X_t is a $n_x \times 1$ vector of model variables; ϵ_t is a $n_\epsilon \times 1$ vector of innovations with $\mathbb{E}(\epsilon_t \epsilon_t') = \Sigma_\epsilon$ with Σ_ϵ being a diagonal matrix and $\mathbb{E}(\epsilon_t \epsilon_{t+h}') = \mathbf{0}_{n_\epsilon \times n_\epsilon}$ for any $h \neq 0$; C_{s_t}, A_{s_t}, and B_{s_t} are conformable matrices that can take on $s_t \in \{1, 2\}$ different values capturing two possible regime realizations in period t that are governed by a Markov transition matrix P. Note that this formulation allows for regime shifts not only in the dynamics of the different variables but also in the variables' trends (e.g., a shift in the inflation target; output growth trend, etc.). A $n_y \times 1$ vector of observables Y_t is then mapped to the vector of model variables as follows[28]

$$Y_t = \Psi_0 + \Psi_1 X_t \tag{20}$$

Proposition 4 provides an expression for the FIRE forecast of X_{t+h}, for any forecast horizon $h > 0$.

PROPOSITION 4. *Given the MSV solution of the model in (19), the rational expectations forecast of X_{t+h} conditional on the full information set available at time t, including the path of regime realization up to period t, is given by*

[28]Adding a vector of measurement errors in (20) would not change any of the results that follow.

$$\mathbb{E}_t X_{t+h} = M_{t,t+h} + Q_{t,t+h} X_t, \tag{21}$$

where matrices $M_{t,t+h}$ and $Q_{t,t+h}$ depend on the regime realized in period t, the transition matrix P, the forecast horizon $h > 0$, as well as matrices A_{s_t}, B_{s_t}, and C_{s_t}.

Proof. See Appendix A.6. □

Combining this expression for the FIRE forecast of X_{t+h} with the measurement equation in (20) then yields the FIRE forecast of the observables' vector,

$$\mathbb{E}_t Y_{t+h} = \Psi_0 + \Psi_1 M_{t,t+h} + \Psi_1 Q_{t,t+h} X_t. \tag{22}$$

.

5.2 Relation of forecast errors with current information

Proposition 5 describes ex-post forecast errors about Y_{t+h} as a function of the vector of current realizations of the endogenous variables of the model, X_t, as well as a function of ex-ante forecast error revisions of the vector of the endogenous variables of the model, $\mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h}$.

PROPOSITION 5. *Given state-space representation (19)-(20) and regime sequence $\{s_t, s_{t+1}, ..., s_{t+h}\}$, ex-post forecast errors under FIRE for any forecasting horizon $h \geq 1$ can be expressed as*

$$Y_{t+h} - \mathbb{E}_t Y_{t+h} = \underbrace{\Theta_{t,t+h}}_{bias} + \underbrace{\Gamma_{t,t+h}}_{predictability} X_t + \xi_{t+h} \tag{23}$$

and

$$Y_{t+h} - E_t Y_{t+h} = \underbrace{\Omega_{t,t+h}}_{bias} + \underbrace{\Delta_{t,t+h}}_{predictability} (\mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h}) + \underbrace{\Lambda_{t-1,t+h}}_{predictability} X_{t-1} + \xi_{t+h}, \tag{24}$$

where $\Theta_{t,t+h}$, $\Omega_{t,t+h}$, $\Gamma_{t,t+h}$, and $\Lambda_{t-1,t+h}$ depend on the ex-post realized regime path $\{s_{t-1}, s_t, s_{t+1}, ..., s_{t+h}\}$, the transition matrix P, forecasting horizon h, as well as matrices A_{s_t}, B_{s_t}, and C_{s_t}; while the error term ξ_{t+h} is uncorrelated with X_t, $(\mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h})$, or X_{t-1}.

Proof. See Appendix A.6. □

Proposition 5 has three important implications. First, it confirms that forecast error bias and ex-post predictability (with respect to current information or ex-ante forecast revisions) are generic features of any FIRE model with regime shifts.

Second, comparison of Proposition 5 with Propositions 1 and 2 makes clear that the relation of forecast errors with current information in the generalized framework differs in two key aspects from the univariate example: (i) ex-post forecast errors in the generalized framework depend on the *entire* vector X_t of available information at the time of forecast and not just on the realization of the variable that is being forecasted; (ii) and matrices $\Gamma_{t,t+h}$ and $\Delta_{t,t+h}$ linking ex-post forecast errors to current information X_t and ex-ante forecast revisions $\mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h}$ are contingent on the *entire* sequence of regime realizations between periods t and $t+h$, $\{s_t, s_{t+1}, ..., s_{t+h}\}$, and not only on the regimes realized in periods t and $t+h$.

Third and as alluded to previously in Section 3, Proposition 5 implies that the reduced-form forecast error regressions considered in the literature will generally be subject to omitted variable bias. Specifically, consider forecast error regressions (1) and (2) for the i^{th} variable in Y; i.e.

$$Y_{i,t+h} - F_t Y_{i,t+h} = \theta + \gamma Y_{it} + e_{t+h} \tag{25}$$

$$Y_{i,t+h} - F_t Y_{i,t+h} = \omega + \delta(F_t Y_{i,t+h} - F_{t-1} Y_{i,t+h}) + e_{t+h} \tag{26}$$

Since $\Gamma_{t,t+h}$ and $\Delta_{t,t+h}$ are generally non-diagonal matrices, Y_{it} and $F_t Y_{i,t+h} - F_{t-1} Y_{i,t+h}$ will not span all the information in X_t, respectively in $\mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h}$ and X_{t-1}, that fully-informed rational agents use to forecast $Y_{i,t+h}$. Consequently, Y_{it} and $F_t Y_{i,t+h} - F_{t-1} Y_{i,t+h}$ will be correlated with e_{t+h}, resulting in omitted variable bias.

This last result provides a potential explanation for the instability of estimates of γ and δ when additional regressors are added, as documented in the literature. Moreover, since $\Delta_{t,t+h} \neq \Gamma_{t,t+h}$, the expected signs of the estimates of γ and δ may differ from each other. In other words, a sufficiently rich FIRE model with regime shifts may, depending on the sequence of regime realizations, generate simultaneously negative OLS estimates of γ and positive OLS estimates of δ, as for example Coibion and Gorodnichenko (2015), Bordalo et al. (2020), Angeletos et al. (2020), Kohlhas and Walther (2021) have found.

Corollary 2 further explores the consequences of regime shifts for three special cases.

Corollary 2. *Proposition 5 nests the following special cases:*

1. *Suppose that there are regime shifts only in the vector of constants; i.e.,* $C_1 \neq C_2$*, but* $A_1 = A_2$ *and* $B_1 = B_2$*. Then,* $\Gamma_{t,t+h} = \Delta_{t,t+h} = \Lambda_{t-1,t+h} = \mathbf{0}_{n_y \times n_x}$ *and* $\Theta_{t,t+h} = \Omega_{t,t+h} \neq \mathbf{0}_{n_y \times 1}$*.*

2. *Now, suppose there are regime shifts only in the relationship between endogenous variables and innovations; i.e.,* $C_1 = C_2$ *and* $A_1 = A_2$*, but* $B_1 \neq B_2$*. Then,* $\Gamma_{t,t+h} = \Delta_{t,t+h} = \Lambda_{t-1,t+h} = \mathbf{0}_{n_y \times n_x}$ *and* $\Theta_{t,t+h} = \Omega_{t,t+h} = \mathbf{0}_{n_y \times 1}$*.*

3. *Finally, suppose there are no regime shifts; i.e.,* $C_1 = C_2$*,* $A_1 = A_2$ *and* $B_1 = B_2$*. Then,* $\Gamma_{t,t+h} = \Delta_{t,t+h} = \Lambda_{t-1,t+h} = \mathbf{0}_{n_y \times n_x}$ *and* $\Theta_{t,t+h} = \mathbf{0}_{n_y \times 1}$*.*

Proof. See Appendix A.7. □

If only the vector of constants is subject to regime shifts, then ex-post forecast errors under FIRE will be biased but not systematically related to current information X_t or to ex-ante forecast revisions $\mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h}$. If regimes apply only to the relationship between the model's endogenous variables and innovations, then forecast errors under FIRE will not exhibit any ex-post predictability.[29] Finally, absent regime shifts, forecast errors under FIRE are not predictable regardless of the complexity of the underlying DGP.

6 Application with a medium-scale DSGE-RS model

We finish by assessing the extent to which a Dynamics Stochastic General Equilibrium model with regime shifts (DSGE-RS) that imposes FIRE and fits macroeconomic dynamics reasonably well is quantitatively consistent with the reduced-form evidence on the predictability of forecast errors discussed in Section 2. The model we consider is a medium-scale New Keynesian model along the lines of Christiano et al. (2005), Smets and Wouters (2007), and Justiniano et al. (2011) augmented with Markov regime shifts in the monetary policy interest rate rule as in Bianchi (2013).[30] We estimate the model with U.S. macroeconomic aggregates using Bayesian likelihood-based techniques and then perform the regime-robust test of FIRE proposed in Section 4.

[29] One can easily show that this result applies if, for instance, one considered Markov regime shifts in the variance of innovations only.

[30] Bianchi (2013) also allows for regime shifts in the volatility of exogenous shocks. Since regime shifts in monetary policy are found to be more important to account for post-WW2 macroeconomic dynamics, we abstract from regime shifts in the volatility of exogenous shocks, although such an extension as well as other regime shifts could in principle be incorporated. See the end of the section for further discussion.

6.1 Model

The economy is populated by a representative household, labor unions, intermediate firms, a final goods producer, and a monetary policy authority. Households maximize a non-separable utility function in goods consumption and labor effort, subject to external habit. Households can save via one-period nominal bonds or investment in physical capital subject to convex adjustment cost. Capital is rented to intermediate firms on a period-by-period basis at a rate that reflects a convex cost of time-varying capital utilization. Household members provide labor to unions that transform labor services into differentiated types and supply them to firms at nominal wages that are subject to Calvo-type infrequent reoptimization. Intermediate firms, in turn, produce differentiated goods with labor and capital and supply the goods to final producers at nominal prices that are subject to Calvo-type infrequent reoptimization. Non-reoptimized nominal wages and prices are partially indexed to lagged inflation. The monetary authority, finally, sets interest rates as a function of lagged interest rates, output growth, inflation, and an exogenous shock. Aside from this monetary policy shock, the economy is also subject to exogenous shocks to the household discount factor, government spending, total factor productivity, investment-specific technology, as well as wage and price markups.

To save on space, we refer the reader to Smets and Wouters (2007) for details and provide a list of log-linearized equilibrium equations in Appendix C.1. The only main difference to the Smets-Wouters model is that we allow the interest rate rule of monetary policy to shift between two regimes; i.e.

$$R_t = \phi^r_{s_t} R_{t-1} + (1 - \phi^r_{s_t}) \left(\phi^{\Delta y}_{s_t} \Delta y_t + \phi^{\pi}_{s_t} \pi_t\right) + v_t, \tag{27}$$

where R_t denotes the nominal short-term interest rate; Δy_t real output growth; π_t inflation (all in deviations from their long-run average values); and v_t is the exogenous monetary policy shock. The response coefficients $\phi^r_{s_t}$, $\phi^{\Delta y}_{s_t}$ and $\phi^{\pi}_{s_t}$ follow an exogenous two-state Markov process $s_t \in \{1, 2\}$ with transition matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \tag{28}$$

where, as in Section 3, $p_{ij} = Pr(s_t = j \mid s_{t-1} = i)$ with $0 < p_{ij} < 1$ and $\sum_{j=1}^{2} p_{ij} = 1$ for both $i, j = 1, 2$.

6.2 Model solution and estimation

We solve the model under FIRE and estimate the parameters, including the Markov transition probabilities, with Bayesian likelihood-based techniques using quarterly U.S. data for output growth, consumption growth, investment growth, real wage growth, labor hours, inflation, and the federal funds rate from 1964:3 to 2020:1. Priors for the different model parameters are set similar to Smets and Wouters (2007), while priors for the monetary policy and regime transition parameters are similar to Bianchi (2013).[31]

Table 2 reports the estimated posterior distribution characteristics for the monetary policy parameters in each of the two regimes as well as the regime transition probabilities. Table 5 and Figure 8 in Appendix C.2 provide information on the prior and posterior distributions for all the other parameters.

Table 2: Posterior distribution estimates of monetary policy rule

	ϕ_1^π	ϕ_2^π	ϕ_1^y	ϕ_2^y	ρ_1	ρ_1	p_{11}	p_{22}
mean	2.63	0.77	0.40	0.62	0.64	0.07	0.85	0.71
mode	2.44	0.81	0.42	0.44	0.61	0.06	0.89	0.60
5%	2.28	0.58	0.20	0.42	0.57	0.02	0.92	0.89
95%	3.00	0.83	0.58	0.88	0.75	0.14	0.75	0.58

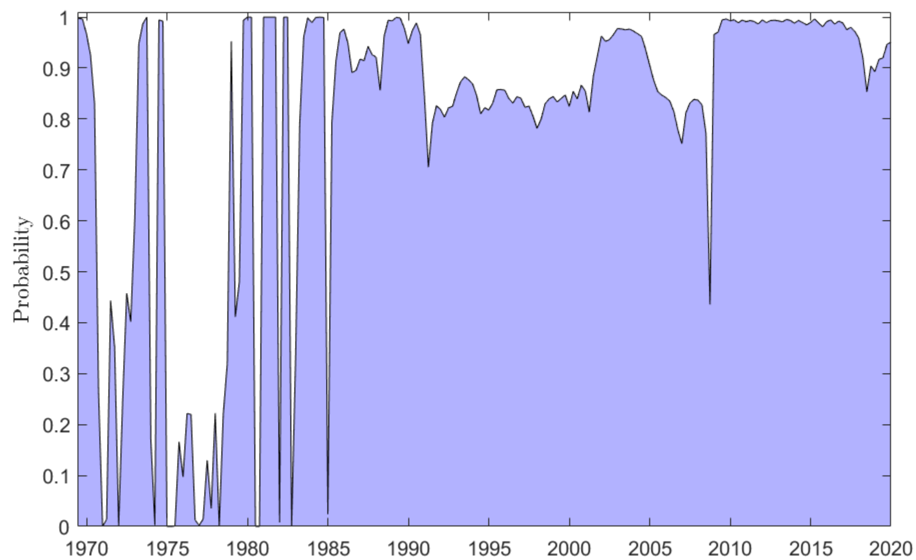
Notes: The table reports the mean, mode, as well as the 5th and 95th percentiles of the posterior distribution of the monetary policy rule parameters in (27) and the Markov transition probabilities in (28), based on 500,000 draws from the Metropolis-Hastings algorithm.

Consistent with Bianchi (2013), we find that one monetary policy regime is more active (regime 1) in the sense that the central bank is estimated to respond aggressively to inflation whereas the other regime is significantly more passive (regime 2). The response of monetary policy to output growth in both regimes is similar, whereas the persistence of nominal interest rates is significantly higher in the more aggressive regime. The aggressive regime is estimated to be highly persistent whereas the passive regime is estimated to be somewhat less persistent.

Figure 6 shows the smoothed regime probabilities for the more aggressive regime as implied by the posterior mode and the data. In line with the common narrative of US monetary history and the results in Bianchi (2013), the estimates imply that monetary policy was primarily in the

[31] We use the RISE Matlab toolbox developed by Maih (2015) to solve and estimate the model. In the case of constant and exogenous transition probabilities, the RISE solution algorithm is similar to Farmer et al. (2011), with the difference that RISE relies on perturbation methods to find the model solution. The estimated posterior distribution is based on the Metropolis-Hasting algorithm with a single chain of 500,000 draws, after discarding 100,000 initial draws. The acceptance rate is about 35%; and the posterior distributions of all the estimated parameters are generally well-behaved. See Appendix C.2 for details.

Figure 6: Probabilities of the aggressive monetary policy regime



Notes: The figure plots the evolution of the smoothed regime probability for the aggressive monetary regime from 1969:3 through 2020:1, evaluated for parameters set at their estimated posterior mode computed.

passive regime (regime 2) during the 1970s and then turned active (regime 1) in the early 1980s during the Volcker years. Active monetary policy continued from the 1980s through the end of the sample, although the probability of a passive regime increases briefly during the 2008-09 Financial Crisis.

6.3 Regime-shift robust test of FIRE

We use the estimated DSGE-RS model to apply the regime-shift robust test of FIRE as described in Section 4. Specifically, we take the estimated DSGE-RS model as the DGP and draw $n = 1, ..., 1000$ parameter vectors Θ^n from the estimated posterior distribution $P(\Theta|\mathcal{Z})$. Then for each Θ^n, we simulate $k = 1, ..., 200$ samples of observations $\left\{y_t^{n,k}\right\}_{t=1}^{T+h}$ and $\left\{\mathbb{E}_t y_{t+h}^{n,k}\right\}_{t=1}^{T}$ conditional on realized regimes $\{s_t^i\}_{t=1}^{T+h}$ drawn from smoothed probabilities $\hat{Pr}(s_t \mid \Theta^n, \mathcal{Z}_T)$, and estimate $\hat{\gamma}_T^{n,k}$ and $\hat{\delta}_T^{n,k}$ for each of these samples. The resulting distributions of simulated $\hat{\gamma}_T^{n,k}$ and $\hat{\delta}_T^{n,k}$ are used to compute p-values for the null that the empirical estimates $\hat{\gamma}_T$ and $\hat{\delta}_T$ were generated by the DSGE-RS model under FIRE. As mentioned earlier, the simulation naturally provides us with the finite sample distribution of the regression coefficients. Hence, we do not need to bias-correct the tests.

Table 3: Regime-robust FIRE test for U.S. inflation and output growth

Panel A: $y_{t+4} - F_t y_{t+4} = \theta + \gamma y_t + e_{t+4}$

	Full sample 1970:2-2019:1				*Subsample 1983:1-2019:1*			
	$\hat{\gamma}_T$	$\mathbb{E}[\hat{\gamma}_T^{FIRE}]$	$\sigma_{\hat{\gamma}_T^{FIRE}}$	$p(FIRE)$	$\hat{\gamma}_T$	$\mathbb{E}[\hat{\gamma}_T^{FIRE}]$	$\sigma_{\hat{\gamma}_T^{FIRE}}$	$p(FIRE)$
Output growth	−0.105	−0.007	0.107	0.359	−0.049	−0.015	0.123	0.778
Inflation	0.049	−0.031	0.093	0.404	−0.169	−0.054	0.110	0.290

Panel B: $y_{t+4} - F_t y_{t+4} = \omega + \delta(F_t y_{t+4} - F_{t-1} y_{t+4}) + e_{t+4}$

	Full sample 1970:2-2019:1				*Subsample 1983:1-2019:1*			
	$\hat{\delta}_T$	$\mathbb{E}[\hat{\delta}_T^{FIRE}]$	$\sigma_{\hat{\delta}_T^{FIRE}}$	$p(FIRE)$	$\hat{\delta}_T$	$\mathbb{E}[\hat{\delta}_T^{FIRE}]$	$\sigma_{\hat{\delta}_T^{FIRE}}$	$p(FIRE)$
Output growth	0.717	−0.022	0.211	0.001	0.507	−0.022	0.130	0.000
Inflation	1.010	−0.017	0.244	0.000	0.111	−0.046	0.145	0.278

Notes: The table reports empirical coefficient estimates of forecast error regressions based on SPF data, the corresponding means and standard deviations of the distribution of coefficient estimates based on simulated data from the DSGE-RS model under FIRE, and the p-value of the null that the empirical coefficient estimates were generated by the DSGE-RS model under FIRE. See the text for details on the simulation process and the construction of the test.

Table 3 reports the empirical estimates $\hat{\gamma}_T$ and $\hat{\delta}_T$ for both the full sample (1970:2-2019:1) and the post-1970 subsample (1983:1-2019:1) from Section 2 together with the corresponding means and standard deviations of the distribution of $\hat{\gamma}_T^{n,k}$ and $\hat{\delta}_T^{n,k}$ simulated from the DSGE-RS model under FIRE, as well as the p-values of the null that the empirical estimates were generated from this DSGE-RS model under FIRE. For both samples, the means of the simulated coefficients estimates $\mathbb{E}[\hat{\gamma}_T^{FIRE}]$ and $\mathbb{E}[\hat{\delta}_T^{FIRE}]$ are close to zero. This is because, similar to the univariate example in Section 4, the effect of regime shifts is relatively small *on average* for both the full sample and the post-1970s subsample. Nevertheless, for the forecast error regressions on current realizations shown in Panel A, the distributions of simulated $\hat{\gamma}_T^{n,k}$ are sufficiently large so that the null of FIRE cannot be rejected at reasonable confidence levels for neither output growth nor inflation. For the forecast error regressions on forecast revisions in Panel B, by contrast, the empirical estimates are generally larger and yield p-values of essentially zero for all but inflation for the post-1970s subsample. Hence, based on these regression estimates, the regime-robust test strongly rejects the null of FIRE.

At the same time, as highlighted in Section 2 and illustrated by the results for inflation in Panel B, the empirical coefficient estimates and with them the outcomes of the test of FIRE can vary importantly over the sample under considerations. To investigate this further and to assess the extent to which our estimated DSGE-RS model generates waves of over-and under-reaction, we return to the rolling window regressions from Figure 1.

6.4 Waves of over- and under-reaction

For each of 40-quarter window from 1970:2 to 2019:1, we compute the 90 percent coverage bands of simulated coefficient estimates from the DSGE-RS model under FIRE as well as the model-implied mean simulated coefficient estimates *conditional on observed data*.[32] We then compare the empirical rolling window estimates from Figure 1 with these model-implied mean estimates and use the coverage bands to assess whether the null of FIRE can be rejected at a reasonable significance level.

Figure 7 reports the results. As shown by the empirical estimates (blue lines) and as already discussed in Section 2, there are large waves of over- and under-predication in the data. For the majority of rolling windows, however, these waves are contained by the 90 percent coverage bands implied by the DSGE-RS model. This is not an artifact of the 40-quarter window length. Even for rolling windows of 60 quarters (15 years), the empirical estimates would be mostly within the coverage bands. Hence, it is generally not possible to reject the null of FIRE at high significance levels.[33]

The model-implied mean estimates (red lines), in turn, display substantial variation across the rolling windows, indicating that the DSGE-RS model is capable of generating waves of over and under-reaction to current information. At the same time, while these waves comove with the waves implied by the empirical estimates for some of the rolling windows, they are overall quite different and on average of smaller magnitude. Hence, the DSGE-RS we use as the DGP to test FIRE falls short of accounting for output growth and inflation forecasts as observed in the SPF data. Given that we do not use SPF data to estimate the model, this shortcoming may not be

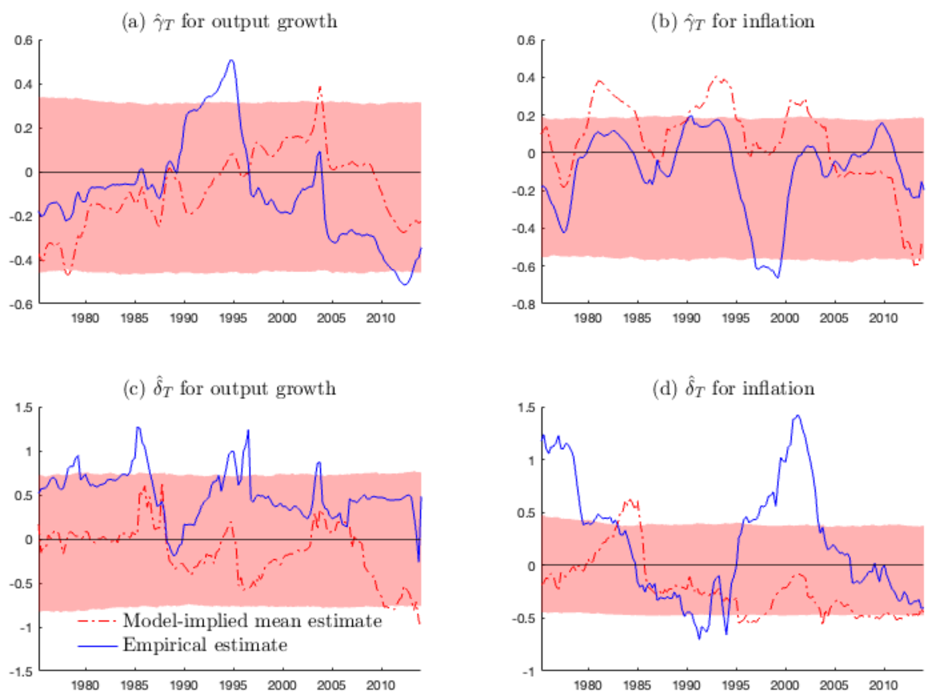
[32]To be more specific, the coverage bands are computed by simulating model-implied data $X_t^{n,k}$ and forecasts $\mathbb{E}_t\left[X_{t+h}^{n,k}\right]$ for each posterior parameter vector Θ^n by drawing $k = 1, ...200$ samples of innovations ϵ_t and states s_t from smoothed probabilities $\hat{Pr}(s_t \mid \Theta^n, \mathcal{Z}_T)$. The coefficient estimates *conditional on observed data*, by contrast, are obtain by simulating $X_t^{n,k}$ and $\mathbb{E}_t\left[X_{t+h}^{n,k}\right]$ from smoothed estimates $\hat{X}_t|s_t;\Theta^n,\mathcal{Z}_T$ and drawing $k = 1, ...200$ samples of states s_t from smoothed probabilities $\hat{Pr}(s_t \mid \Theta^n, \mathcal{Z}_T)$. These conditional values should be interpreted as resulting from a particular draw of innovations ϵ_t that represents the best model-implied estimate given the sample period under consideration.

[33]Figure 10 in Appendix C.4 reports the p-values of the FIRE test for each of the rolling window estimates.

all that surprising.[34] Nonetheless, it means that the chosen DGP fails to capture the dynamics of expectations as observed in the data.

Also note that the model-implied coverage bands barely move and, likewise, $\mathbb{E}[\hat{\gamma}_T^{FIRE}]$ and $\mathbb{E}[\hat{\delta}_T^{FIRE}]$ (not shown) deviate little from zero across the rolling windows. This means that the simulated regression coefficients implied by the model are, on average, not sensitive to variations in the sequence of realized regimes (i.e., the $f_{11}^{(h)}$ and $f_{22}^{(h)}$ across the rolling window samples) relative to agents expectations (i.e., the $p_{11}^{(h)}$ and $p_{22}^{(h)}$ implied by the model's Markov transition matrix P).

Figure 7: Regime-robust FIRE test across subsamples



Notes: The figure shows 40-quarter rolling regression estimates of $\hat{\gamma}_t$ and $\hat{\delta}_t$ in (1) and (2) based on SPF data (blue solid line) and based on data simulated from the DSGE-RS model (red dash-dotted line). The shaded areas in red show the 90% coverage bands of the coefficients estimates implied by the DSGE-RS model. See main text for details. The estimates are centered at the midpoint of the rolling regression window (e.g. 1980 denotes the regression window 1975:1 to 1984:4).

6.5 Taking stock

The application with the medium-scale DSGE-RS model yields two main lessons. First, our regime-robust test fails to reject FIRE decisively. Second, while the model generates sizable waves of over-

[34] While we could add SPF or other forecast data to estimate the model, this would require us to either drop some of the macro aggregates used currently in the estimation or add more shocks so as to avoid stochastic singularity.

and under-reaction of expectations to current information, regime shifts in monetary policy play only a small role for these waves, and the waves are generally quite different from the empirical estimates. We do not see this as a negative result about the potential of regime shifts to account for the large and time-varying waves in forecast error predictability. Indeed, it is worth remembering that the simulations are conditional on the particular DSGE-RS model used as DGP, and that for other DGPs (including the univariate model used in Sections 3 and 4), the effect of variations in regime realizations can be substantially larger. Instead, we consider the waves in forecast error predictability as a challenge for this DSGE-RS model, which is generally considered a benchmark for modern business cycle analysis and our understanding of monetary policy. This provides empirical motivation to assess the extent to which alternative DGPs with other types of regime shifts (e.g., in trend growth as in Foerster and Matthes, 2022 or trend inflation) as well as potential departures from FIRE are capable of generating the expectations dynamics observed in the data.

7 Conclusion

The present paper shows that regime shifts in FIRE models lead to predictable, regime-dependent ex-post forecast errors. In general, in the presence of regime shifts, forecast errors become a complicated function of the current state of the economy and the sequence of realized regimes over the entire forecast horizon. This implies that reduced-form forecast error regressions by themselves are not informative about alternative theories of expectations formation. Furthermore, regime shifts imply that expectations exhibit waves of over- and under-reaction to current information in rolling sample windows. Using survey-based forecast data of inflation and output growth constructed from the SPF, we confirm the existence of such waves.

Based on these insights, we propose a regime-robust test of FIRE and implement it on SPF data for output growth and inflation conditional on an estimated medium-scale DSGE model with regime shifts in the monetary policy interest rate rule. Conditional on this data-generating process, we cannot reject decisively that the observed waves of over- and under-reaction were generated under FIRE. This should be taken as neither an endorsement of FIRE nor a dismissal of alternative theories of expectations formation. Indeed, there is much empirical evidence that even relatively sophisticated market participants are subject to imperfect information and make decisions that are hard to square with the assumption of rational expectations. Furthermore, the model we use as data-generating process generates waves of over- and under-prediction that generally quite different from the ones we observe in the data. We view this as empirical motivation to consider models with alternative types of regime shifts as well as potential departures from FIRE to account for the dynamics of average expectations observed in the data.

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Appendix

A Proofs

A.1 Proof of Proposition 1

The forecasting errors about y_{t+h}, for any $h>0$, are given by

$$\begin{aligned} FE_{t,t+h} &= y_{t+h} - \mathbb{E}_t y_{t+h} \\ &= a_{s_{t+h}}\phi^h x_t - P^{(h)}_{s_t:}\boldsymbol{a}\phi^h x_t + a_{s_{t+h}}\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau} \\ &= \left(a_{s_{t+h}} - P^{(h)}_{s_t:}\boldsymbol{a}\right)\phi^h x_t + a_{s_{t+h}}\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau} \\ &= (a_{s_{t+h}} - p^{(h)}_{s_t,1}a_1 - p^{(h)}_{s_t,2}a_2)\phi^h x_t + a_{s_{t+h}}\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau} \\ &= \begin{cases} \left((1 - p^{(h)}_{s_t,1}a_1 - p^{(h)}_{s_t,2}a_2\right)\phi^h x_t + a_1\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau} & \text{if } s_{t+h}=1 \\ -p^{(h)}_{s_t,1}(a_1-a_2)\phi^h x_t + a_2\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau} & \text{if } s_{t+h}=2 \end{cases} \end{aligned} \tag{A.1}$$

where the last equality follows from $p^{(h)}_{s_t,1} + p^{(h)}_{s_t,2} = 1$. We show this through proof by induction. Clearly, $P_1(s_t,1)+P_1(s_t,2) = 1$ for any $s_t \in \{1,2\}$. Suppose that $p^{(h)}_{s_t,1}+p^{(h)}_{s_t,2} = 1$. We should prove that $p^{(h+1)}_{s_t,1} + p^{(h+1)}_{s_t,2} = 1$. We have that

$$P^{(h+1)} = PP^{(h)} = \begin{bmatrix} p_{11}p^{(h)}_{11} + p_{12}p^{(h)}_{21} & p_{11}p^{(h)}_{12} + p_{12}p^{(h)}_{22} \\ p_{21}p^{(h)}_{11} + p_{22}p^{(h)}_{21} & p_{21}p^{(h)}_{12} + p_{22}p^{(h)}_{22} \end{bmatrix} \tag{A.2}$$

Then, $p_{11}p^{(h)}_{11} + p_{12}p^{(h)}_{21} + p_{11}p^{(h)}_{12} + p_{12}p^{(h)}_{22} = p_{11}+p_{12} = 1$ and $p_{21}p^{(h)}_{11} + p_{22}p^{(h)}_{21} + p_{21}p^{(h)}_{12} + p_{22}p^{(h)}_{22} = p_{21}+p_{22} = 1$. Note that $p^{(h)}_{s_t,1} = 1-p^{(h)}_{s_t,2}$, i.e., $p^{(h)}_{s_t,1} = 1-p^{(h)}_{s_t,s_{t+h}}$ for $s_{t+h}=2$, and $x_t = \frac{y_t}{a_{s_t}}$, therefore,

$$FE_{t,t+h} = \frac{(-1)^{s_{t+h}-1}(1-p^{(h)}_{s_t,s_{t+h}})(a_1-a_2)\phi^h}{a_{s_t}}x_t + a_{s_{t+h}}\sum_{\tau=1}^{h}\phi^{h-\tau}\varepsilon_{t+\tau} \tag{A.3}$$

A.2 Proof of Corollary 1

Consider

$$a_{s_{t+h}} - P^{(h)}_{s_t:}\boldsymbol{a} = \begin{cases} (1-p^{(h)}_{s_t,1})(a_1-a_2) & \text{if } s_{t+h}=1 \\ -(1-p^{(h)}_{s_t,2})(a_1-a_2) & \text{if } s_{t+h}=2 \end{cases}$$

One can show via proof by induction that for $0 < p_{11}, p_{22} < 1$, it follows that $0 < p^{(h)}_{s_t,s_{t+h}} < 1$ for any $s_t, s_{t+h} \in \{1,2\}$. Clearly, $1 - p_{s_t,s_{t+h}} > 0$ for any $s_t \in \{1,2\}$. Suppose that $1 - p^{(h)}_{s_t,s_{t+h}} > 0$; we should prove that $1 - p^{(h+1)}_{s_t,s_{t+h}} > 0$. Consider

$$P^{(h+1)} = PP^{(h)} = \begin{bmatrix} p_{11}p^{(h)}_{11} + p_{12}p^{(h)}_{21} & p_{11}p^{(h)}_{12} + p_{12}p^{(h)}_{22} \\ p_{21}p^{(h)}_{11} + p_{22}p^{(h)}_{21} & p_{21}p^{(h)}_{12} + p_{22}p^{(h)}_{22} \end{bmatrix} \tag{A.4}$$

Then,

$$1-p_{11}p^{(h)}_{11}-p_{12}p^{(h)}_{21} = 1-p_{11}(p^{(h)}_{11}-p^{(h)}_{21})-p^{(h)}_{21} = p^{(h)}_{22}-p_{11}(p^{(h)}_{11}+p^{(h)}_{22}-1) = p^{(h)}_{22}(1-p_{11})+p_{11}(1-p^{(h)}_{11}) > 0$$

$$1 - p_{11}p^{(h)}_{12} - p_{12}p^{(h)}_{22} = 1 - p_{11}(1 - p^{(h)}_{11} - p^{(h)}_{22}) - p^{(h)}_{22} = p^{(h)}_{21}(1-p_{11}) + p_{11}p^{(h)}_{11} > 0$$

$$1 - p_{21}p^{(h)}_{11} - p_{22}p^{(h)}_{21} = 1 - p_{21}(p^{(h)}_{11} - 1 + p^{(h)}_{22}) - p^{(h)}_{21} = p^{(h)}_{22}p_{22} + p_{21}(1 - p^{(h)}_{11}) > 0$$

$$1 - p_{21}p^{(h)}_{12} - p_{22}p^{(h)}_{22} = 1 - p_{21}(p^{(h)}_{12} - 1 + p^{(h)}_{21}) - p^{(h)}_{22} = p^{(h)}_{21}p_{11} + p_{12}p^{(h)}_{11} > 0$$

Hence,

$$sign(a_{s_{t+h}} - P^{(h)}_{s_t:}\boldsymbol{a}) = \begin{cases} sign(a_1-a_2) & \text{if } s_{t+h}=1 \\ -sign(a_1-a_2) & \text{if } s_{t+h}=2 \end{cases}$$

A.3 Proof of Proposition 2

Forecast revisions can be expressed as

$$\mathbb{E}_t y_{t+h} - \mathbb{E}_{t-1}y_{t+h} = \phi^h \frac{P^{(h)}_{s_t:}\boldsymbol{a}}{a_{s_t}} y_t - \phi^{h+1}\frac{P^{(h+1)}_{s_{t-1}:}\boldsymbol{a}}{a_{s_{t-1}}} y_{t-1}. \tag{A.5}$$

To obtain (14), we isolate y_t from (A.5), and substitute for it into (12).

A.4 Proof of Proposition 3

The expected OLS coefficient estimate of regression $y_{t+h} - F_t y_{t+h} = \gamma y_t + e_{t+h}$ conditional on regime sequence $\{s_t\}_{t=1}^{T+h}$ is

$$\mathbb{E}\left[\hat{\gamma}_T | \{s_t\}_{t=1}^{T+h}\right] = \mathbb{E}\left[\sum_{t=1}^{T} \frac{y_t}{\sum_{t=1}^{T} y_t^2}(y_{t+h} - F_t y_{t+h}) | \{s_t\}_{t=1}^{T+h}\right].$$

Assuming that the data is generated by the univariate regime-switching model under FIRE in (8)-(10), we can use Proposition 1 to express

$$\mathbb{E}\left[\hat{\gamma}_T | \{s_t\}_{t=1}^{T+h}\right] = \mathbb{E}\left[\sum_{t=1}^{T} \frac{y_t}{\sum_{t=1}^{T} y_t^2}\gamma_{s_t,s_{t+h}}^{(h)} y_t | \{s_t\}_{t=1}^{T+h}\right] + \mathbb{E}\left[\sum_{t=1}^{T} \frac{y_t}{\sum_{t=1}^{T} y_t^2}\xi_{t+h} | \{s_t\}_{t=1}^{T+h}\right], \quad \text{(A.6)}$$

where $\gamma_{s_t,s_{t+h}}^{(h)} \equiv \frac{(-1)^{s_{t+h}-1}(a_1-a_2)\left(1-P_{s_t,s_{t+h}}^{(h)}\right)\phi^h}{a_{s_t}}$ and $\xi_{t+h} \equiv a_{s_{t+h}} \sum_{\tau=1}^{h} \phi^{h-\tau}\varepsilon_{t+\tau}$. Further remembering that $y_t = a_{s_t} x_t$, we can write

$$\mathbb{E}\left[\hat{\gamma}_T | \{s_t\}_{t=1}^{T+h}\right] = \mathbb{E}\left[\sum_{t=1}^{T} \frac{\gamma_{s_t,s_{t+h}}^{(h)} a_{s_t}^2 x_t^2}{\sum_{t=1}^{T} a_{s_t}^2 x_t^2} | \{s_t\}_{t=1}^{T+h}\right] + \mathbb{E}\left[\sum_{t=1}^{T} \frac{a_{s_t} x_t}{\sum_{t=1}^{T} a_{s_t}^2 x_t^2}\xi_{t+h} | \{s_t\}_{t=1}^{T+h}\right]. \quad \text{(A.7)}$$

The first term of this expression is the expected estimate implied by the data-generating process while the second term is the bias arising from the fact that in finite samples, innovations $\varepsilon_{t+1}, \varepsilon_{t+2}, ..., \varepsilon_{t+h}$ affect not only ξ_{t+h} but also $\sum_{t=1}^{T} a_{s_t}^2 x_t^2$. In particular, for positive values of a_1, a_2 and ϕ, this bias is negative.

Conditional on sufficiently persistent regime realizations, the numerator and the denominator of the first term are close to independent of each other. Hence, the following approximation provides a good characterization of the expected bias-corrected estimate $\hat{\gamma}_T^c$

$$\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] = \mathbb{E}\left[\sum_{t=1}^{T} \frac{\gamma_{s_t,s_{t+h}}^{(h)} a_{s_t}^2 x_t^2}{\sum_{t=1}^{T} a_{s_t}^2 x_t^2} | \{s_t\}_{t=1}^{T+h}\right] \approx \frac{\mathbb{E}\left[\left(\sum_{t=1}^{T} \gamma_{s_t,s_{t+h}}^{(h)} a_{s_t}^2 x_t^2\right) | \{s_t\}_{t=1}^{T+h}\right]}{\mathbb{E}\left[\sum_{t=1}^{T} a_{s_t}^2 x_t^2 | \{s_t\}_{t=1}^{T+h}\right]}$$
$$= \frac{\sum_{t=1}^{T} \gamma_{s_t,s_{t+h}}^{(h)} a_{s_t}^2 \mathbb{E}\left[x_t^2\right]}{\sum_{t=1}^{T} a_{s_t}^2 \mathbb{E}\left[x_t^2\right]} = \frac{\sum_{j=1}^{2}\sum_{i=1}^{2} a_i^2 \gamma_{ij}^{(h)} \mathcal{F}_T^{(h)}(i,j)}{\sum_{i=1}^{2} a_i^2 \mathcal{F}_T^{(h)}(i)}, \quad \text{(A.8)}$$

where the last equation makes use of the fact that under the assumption of two regimes, $\gamma_{s_t,s_{t+h}}^{(h)} a_{s_t}^2$ takes on one of four values (each with joint sample frequency $\mathcal{F}_T^{(h)}(i,j)$) while $a_{s_t}^2$ takes on one of two values (each with sample frequency $\mathcal{F}_T^{(h)}(i)$).

To analyze the properties of $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right]$, define the following conditional sample transition probabilities:

$$f_{ji}^{(h)} = \frac{\sum_{t=1}^T \mathbf{1}(s_t = i, s_{t+h} = j)}{\sum_{t=1}^T \mathbf{1}(s_{t+h} = j)} = \frac{\mathcal{F}_T^{(h)}(i,j)}{\mathcal{F}_T^{(h)}(j)}$$

where $\sum_{i=1}^2 f_{ji}^{(h)} = 1$. One can show that

$$\mathcal{F}_T^{(h)}(j) = \frac{1}{T}\sum_{t=1}^T \mathbf{1}(s_{t+h} = j) \approx \begin{cases} \frac{1-f_{22}^{(h)}}{2-f_{11}^{(h)}-f_{22}^{(h)}} & \text{if } j = 1 \\ \\ \frac{1-f_{11}^{(h)}}{2-f_{11}^{(h)}-f_{22}^{(h)}} & \text{if } j = 2 \end{cases}$$

Hence, $\mathcal{F}_T^{(h)}(i,j) = f_{ji}^{(h)}\mathcal{F}_T^{(h)}(j)$, and $\mathcal{F}_T^{(h)}(j)$ depends on $f_{11}^{(h)}$and $f_{22}^{(h)}$only. Substituting these expressions together with $\gamma_{s_t,s_{t+h}}^{(h)} \equiv \frac{(-1)^{s_{t+h}-1}(a_1-a_2)\left(1-P_{s_t,s_{t+h}}^{(h)}\right)\phi^h}{a_{s_t}}$ in the above expression $\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right]$, we obtain

$$\mathbb{E}\left[\hat{\gamma}_T^c | \{s_t\}_{t=1}^{T+h}\right] = \underbrace{\frac{\phi^h(a_1-a_2)}{a_1^2(1-f_{22}^{(h)}) + a_2^2(1-f_{11}^{(h)})}}_{(+)} \underbrace{[a_1(1-f_{22}^{(h)})(f_{11}^{(h)}-p_{11}^{(h)}) - a_2(1-f_{11}^{(h)})(f_{22}^{(h)}-p_{22}^{(h)})]}_{g(f_{11}^{(h)},f_{22}^{(h)})} \tag{A.9}$$

Clearly, the sign of $\mathbb{E}[\gamma|\{s_t\}_{t=1}^{T+h}]$ depends on the sign of $g(f_{11}^{(h)}, f_{22}^{(h)})$. The frontier in the $(f_{11}^{(h)}, f_{22}^{(h)})$ plane for which is given by

$$f_{11}^{(h)} = g(f_{22}^{(h)}) = \frac{a_1 p_{11}^{(h)}(1-f_{22}^{(h)}) - a_2(p_{22}^{(h)} - f_{22}^{(h)})}{a_1(1-f_{22}^{(h)}) - a_2(p_{22}^{(h)} - f_{22}^{(h)})}$$

where $a_1(1-f_{22}^{(h)}) > a_2(p_{22}^{(h)}-f_{22}^{(h)})$ for any $0 \le f_{22}^{(h)} \le 1$, given that $a_1 > a_2$. Moreover, $g(p_{22}^{(h)}) = p_{11}^{(h)}$ and $g(1) = 1$. Then, $\mathbb{E}[\gamma|\{s_t\}_{t=1}^{T+h}] \lesseqgtr 0 \iff f_{11}^{(h)} \lesseqgtr g(f_{22}^{(h)})$.

A.5 Proof of Proposition 4

Full-information rational expectations about X_{t+h} are given by:

$$
\begin{aligned}
\mathbb{E}_t X_{t+h} &= \mathbb{E}_t \left(C_{t+h} + A_{t+h} X_{t+h-1} + B_{t+h}\epsilon_{t+h}\right) \\
&= \mathbb{E}_t \left(C_{t+h} + \sum_{\tau=1}^{h-1} \left(\prod_{l=0}^{\tau} A_{t+h-\tau} \right) C_{t+h-\tau} + \prod_{\tau=0}^{h-1} A_{t+h-\tau} X_t \right)
\end{aligned}
$$

We will show through a proof by induction that

$$
\begin{aligned}
\mathbb{E}_t X_{t+h} &= \left(\begin{bmatrix} C_1 & C_2 \end{bmatrix} (P')^h + \begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \sum_{\tau=1}^{h-1} \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{\tau-1} \widetilde{C}(P')^{h-\tau} \right) I_{:s_t} \\
&+ \begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{h-1} \bar{\iota}_{s_t} X_t
\end{aligned}
$$

where $\widetilde{C} = \begin{bmatrix} C_1 & 0_{n_x \times 1} \\ 0_{n_x \times 1} & C_2 \end{bmatrix}$, $\widetilde{A} = \begin{bmatrix} A_1 & 0_{n_x \times n_x} \\ 0_{n_x \times n_x} & A_2 \end{bmatrix}$, $\bar{\iota}_{s_t}$ is a $2n_x \times n_x$ size matrix whose s_t^{th} block of n_x rows together with the columns form an identity matrix and the rest of elements are 0,and $I_{:s_t}$is the s_t^{th} column of a 2×2 identity matrix.

One can check that the expression above holds for $h \in \{1, 2\}$. Suppose it also holds for $h = \bar{h}$; does it also apply for $h = \bar{h} + 1$?

$$
\begin{aligned}
\mathbb{E}_t X_{t+\bar{h}+1} &= \mathbb{E}_t \left(C_{t+\bar{h}+1} + A_{t+\bar{h}+1} X_{t+\bar{h}}\right) = \mathbb{E}_t \left[\mathbb{E}_{t+1} \left(C_{t+\bar{h}+1} + A_{t+\bar{h}+1} X_{t+\bar{h}}\right)\right] \\
&= \mathbb{E}_t \left(\begin{bmatrix} C_1 & C_2 \end{bmatrix} (P')^{\bar{h}} + \begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \sum_{\tau=1}^{\bar{h}-1} \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{\tau-1} \widetilde{C}(P')^{\bar{h}-\tau} \right) I_{:s_{t+1}} \\
&+ \mathbb{E}_t \left(\begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{\bar{h}-1} \bar{\iota}_{s_{t+1}} X_{t+1} \right) \\
&= \left(\begin{bmatrix} C_1 & C_2 \end{bmatrix} (P')^{\bar{h}} + \begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \sum_{\tau=1}^{\bar{h}-1} \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{\tau-1} \widetilde{C}(P')^{\bar{h}-\tau} \right) \mathbb{E}_t I_{:s_{t+1}} \\
&+ \begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{\bar{h}-1} \mathbb{E}_t \left(\bar{\iota}_{s_{t+1}} A_{t+1} X_t + \bar{\iota}_{s_{t+1}} C_{t+1} \right) \\
&= \begin{bmatrix} C_1 & C_2 \end{bmatrix} (P')^{\bar{h}+1} I_{:s_t} + \begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \sum_{\tau=1}^{\bar{h}} \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{\tau-1} \widetilde{C}(P')^{\bar{h}+1-\tau} I_{:s_t} \\
&+ \underbrace{\begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{\bar{h}} \bar{\iota}_{s_t}}_{\text{response to info at time t}} X_t
\end{aligned}
$$

where the equality in the 7 row follows from $\mathbb{E}_t I_{:s_{t+1}} = P' I_{:s_t}$. Therefore,

$$M_{t,t+h} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} (P')^h I_{:s_t} + \begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \sum_{\tau=1}^{h-1} \left(\widetilde{A}(P' \otimes I_{n_x})\right)^{\tau-1} \widetilde{C}(P')^{h-\tau} I_{:s_t} \tag{A.10}$$

$$Q_{t,t+h} = \begin{bmatrix} A_1 & A_2 \end{bmatrix} (P' \otimes I_{n_x}) \left(\widetilde{A}(P' \otimes I_{n_x})\right)^{h-1} \bar{\iota}_{s_t} \tag{A.11}$$

Finally, we note that $M_{t,t+1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} P' I_{:s_t}$.

A.6 Proof of Proposition 5

Consider the ex-post forecasting error about vector Y_{t+h}, where $h \geq 1$:

$$\begin{aligned} FE_{t,t+h} &= Y_{t+h} - \mathbb{E}_t Y_{t+h} \\ &= \Psi_1 \left(C_{s_{t+h}} + A_{s_{t+h}} X_{t+h-1} + B_{t+h} \epsilon_{t+h} - M_{t,t+h} - Q_{t,t+h} X_t \right) \\ &= \Psi_1 C_{s_{t+h}} + \Psi_1 A_{s_{t+h}} \left(C_{s_{t+h-1}} + A_{s+h-1} X_{t+h-1} + B_{s_{t+h-1}} \epsilon_{t+h-1} - M_{t,t+h} - Q_{t,t+h} X_t \right) + \Psi_1 B_{s_{t+h}} \epsilon_{t+h} \\ &= \ldots \\ &= \underbrace{\Psi_1 \left(C_{s_{t+h}} + \sum_{\tau=1}^{h-1} \left(\prod_{l=0}^{\tau} A_{s_{t+h-l}} \right) C_{s_{t+h-\tau}} - M_{t,t+h} \right)}_{\Theta_{t,t+h} \equiv \text{bias}} + \underbrace{\Psi_1 \left(\prod_{\tau=1}^{h} A_{s_{t+\tau}} - Q_{t,t+h} \right)}_{\Gamma_{t,t+h} \equiv \text{ex-post predictability}} X_t + error_{t+h} \end{aligned} \tag{A.12}$$

where $error_{t+h} = \Psi_1 \sum_{\tau=1}^{h-1} (\prod_{l=0}^{\tau} A_{s_{t+h-l}}) B_{s_{t+h-\tau}} \epsilon_{t+h-\tau} + \Psi_1 B_{s_{t+h}} \epsilon_{t+h}$. We now turn to expressing ex-post forecast errors as a function of ex-ante forecast revisions. The FIRE forecasts about the endogenous variables vector X_{t+h} in periods t and $(t-1)$ are given by, respectively,

$$\mathbb{E}_t X_{t+h} = M_{t,t+h} + Q_{t,t+h} X_t \tag{A.13}$$

$$\mathbb{E}_{t-1} X_{t+h} = M_{t-1,t+h} + Q_{t-1,t+h} X_{t-1} \tag{A.14}$$

Hence, the ex-ante forecast revision about X_{t+h} is given by

$$FR_{t,t+h} = \mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h} = M_{t,t+h} - M_{t-1,t+h} + Q_{t,t+h} X_t - Q_{t-1,t+h} X_{t-1} \tag{A.15}$$

If $Q_{t,t+h}$ is invertible, then from (A.15), $X_t = Q^{-1}_{t,t+h}(\mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h}) + Q^{-1}_{t,t+h} Q_{t-1,t+h} X_{t-1} - Q^{-1}_{t,t+h}(M_{t,t+h} - M_{t-1,t+h})$. Substituting for X_t into (A.12), we can rewrite ex-post forecast errors as a function of ex-ante forecast revisions.

$$Y_{t+h} - \mathbb{E}_t Y_{t+h} = \underbrace{\big(\Theta_{t,t+h} - \Gamma_{t,t+h} Q^{-1}_{t,t+h}(M_{t,t+h} - M_{t-1,t+h})\big)}_{=\Omega_{t,t+h}} + \underbrace{\Gamma_{t,t+h} Q^{-1}_{t,t+h}}_{=\Delta_{t,t+h}}(\mathbb{E}_t X_{t+h} - \mathbb{E}_{t-1} X_{t+h}) \tag{A.16}$$

$$+ \underbrace{\Gamma_{t,t+h} Q^{-1}_{t,t+h} Q_{t-1,t+h}}_{=\Lambda_{t-1,t+h}} X_{t-1} + error_{t+h}$$

If $Q_{t,t+h}$ is non-invertible, we proceed as follows. Let q_{ij} and $\widetilde{q}_{ij}$ denote the element located in row i and column j in matrices $Q_{t,t+h}$ and $\widetilde{Q}_{t-1,t+h}$, respectively. Furthermore, let m_i be the element located in row i in matrix $(M_{t,t+h} - M_{t-1,t+h})$. The ex-ante forecast revision of any variable X_i in X can be written as:

$$FR_{i,t,t+h} = \mathbb{E}_t X_{i,t+h} - \mathbb{E}_{t-1} X_{i,t+h} = m_i + \sum_{j=1}^{n_x} q_{ij} X_{j,t} - \sum_{j=1}^{n_x} \widetilde{q}_{ij} X_{j,t-1} \tag{A.17}$$

Then, any variable X_{kt} in X_t can be written as a function of the ex-ante forecast revision about variable $X_{i,t+h}$, where i is chosen such that $q_{ik} \neq 0$, as well as X_{-kt}, X_{t-1}, and a constant:

$$\begin{aligned} X_{kt} &= \frac{FR_{i,t,t+h} - \sum_{j\neq k} q_{ij} X_{jt} + \sum_j \widetilde{q}_{ij} X_{j,t-1} - m_i}{q_{ik}} \qquad \text{(A.18)} \\ &= \underbrace{\begin{bmatrix} 0 & 0 & ... & \frac{1}{q_{ik}} & ... & 0 & 0 \end{bmatrix}}_{Q^{-}(k,:)} FR_{t,t+h} - \underbrace{\begin{bmatrix} \frac{q_{i1}}{q_{ik}} & ... & \frac{q_{i,k-1}}{q_{ik}} & 0 & \frac{q_{i,k+1}}{q_{ik}} & ... & \frac{q_{in_x}}{q_{ik}} \end{bmatrix}}_{Q_Q(k,:)} X_t + \underbrace{\begin{bmatrix} \frac{\widetilde{q}_{i1}}{q_{ik}} & ... & \frac{\widetilde{q}_{in_x}}{q_{ik}} \end{bmatrix}}_{\widetilde{Q}_Q(k,:)} X_{t-1} \\ &- \underbrace{\frac{m_i}{q_{ik}}}_{M_{Q(k,:)}} \end{aligned}$$

It follows that vector X_t can be written as a function of ex-ante forecast revisions as described below:

$$X_t = Q^{-}FR_{t,t+h} - Q_Q X_t + \widetilde{Q}_Q X_{t-1} - M_Q$$

From here, we have that $X_t = (I_{nx} + Q_Q)^{-1}(Q^{-}FR_{t,t+h} + \widetilde{Q}_Q X_{t-1} - M_Q)$, and that

$$\begin{aligned} Y_{t+h} - \mathbb{E}_t Y_{t+h} &= \underbrace{\Theta_{t,t+h} - \Gamma_{t,t+h}(I_{nx} + Q_Q)^{-1} M_Q}_{=\Omega_{t,t+h}} + \underbrace{\Gamma_{t,t+h}(I_{nx} + Q_Q)^{-1} Q^{-}}_{=\Delta_{t,t+h}} FR_{t,t+h} \quad \text{(A.19)} \\ &+ \underbrace{\Gamma_{t,t+h}(I_{nx} + Q_Q)^{-1} \widetilde{Q}_Q}_{=\Lambda_{t-1,t+h}} X_{t-1} + error_{t+h} \end{aligned}$$

A.7 Proof of Corollary 2

1. Consider the case when $C_1 \neq C_2$, while $A_1 = A_2 = A$ and $B_1 = B_2 = B$. Note that, for $A_1 = A_2 = A$,the following is true:

$$Q_{t,t+h} = \Psi_1 \begin{bmatrix} A & A \end{bmatrix} (P' \otimes I_{n_x}) \left(\widetilde{A}(P' \otimes I_{n_x}) \right)^{h-1} \bar{\iota}_{s_t} = \Psi_1 A^h \quad \text{(A.20)}$$

   Therefore, $\Gamma_{t,t+h} = \Psi_1(A^h - A^h) = \mathbf{0}_{n_y \times n_x}$, and, given that $B_1 = B_2 = B$, we have that $error_{t+h} = \Psi_1 \sum_{\tau=0}^{h-1} A^\tau B \epsilon_{t+h-\tau}$. Furthermore,

$$M_{t,t+h} = \Psi_1 \sum_{\tau=0}^{h-1} A^\tau \widetilde{C} (P')^{h-\tau} I_{:s_t} \neq \mathbf{0}_{n_y \times 1} \quad \text{(A.21)}$$

   In this case, $\Theta_{t,t+h} = \Psi_1 \sum_{\tau=0}^{h-1} A^\tau (C_{s_{t+h-\tau}} - \widetilde{C}(P')^{h-\tau} I_{:s_t}) \neq \mathbf{0}_{n_y \times 1}$,therefore, ex-post forecast errors will be biased, but they will not respond to information embedded in X_t.

2. Now, suppose that $C_1 = C_2 = C$ and $A_1 = A_2 = A$, while $B_1 \neq B_2$. Note that, given that $C_1 = C_2 = C$,the following is true:

$$M_{t,t+h} = \Psi_1 \sum_{\tau=0}^{h-1} A^\tau C \quad \text{(A.22)}$$

   So, $\Theta_{t,t+h} = \mathbf{0}_{n_y \times n_x}$, implying that ex-post forecast error are not biased. Furthermore, $A_1 = A_2 = A$ implies that $\Gamma_{t,t+h} = \mathbf{0}_{n_y \times n_x}$. For $B_1 \neq B_2$, the error term is as defined in Section A.6. Consequently, when the regime shifts affect only the relationship between the endogenous variables and innovations, ex-forecast errors are just accumulated noise, similar to the case of no regime shifts discussed below.

3. Finally, shutting down all regime shifts in the model implies that $\Theta_{t,t+h} = \mathbf{0}_{n_y \times 1}$, $\Gamma_{t,t+h} = \mathbf{0}_{n_y \times n_x}$, and $error_{t+h} = \Omega \sum_{\tau=0}^{h-1} A^{\tau} B \epsilon_{t+h-\tau}$. Hence, in this case, forecast errors are accumulated noise similar to the second case.

B Univariate model

Table 4 reports the characteristics of the prior and posterior distributions for the univariate model.

Table 4: Prior and posterior distribution for the model with regime shifts

	Prior			Posterior		
	pdf	5%	95%	mean	5%	95%
a_1	$\mathcal{U}$	0.1	5	4.17	2.58	5.27
a_2	$\mathcal{U}$	0.1	5	1.44	0.86	1.98
ϕ	$\mathcal{B}$	0.2	0.8	0.87	0.82	0.92
σ	$\mathcal{IG}$	0.01	2	0.38	0.26	0.58
p_{12}	$\mathcal{B}$	0.01	0.05	0.03	0.01	0.04
p_{21}	$\mathcal{B}$	0.01	0.05	0.02	0.01	0.04

C Medium-scale model

C.1 Description of the log-linearized model

We briefly describe the medium-scale model which largely extracts from Smets and Wouters (2007), and refer the reader to their paper for details on the model's micro-foundations. The aggregate resource constraint is given by

$$y_t = c_y c_t + i_y i_t + z_y z_t + e_t^g \tag{C.1}$$

where y_t is output, c_t consumption, z_t capital utilization rate, e_t^g exogenous government spending such that $e_t^g = \rho_g e_{t-1}^g + \varepsilon_t^g + \rho_{ga}\varepsilon_t^a$, with $\varepsilon_t^j \sim \mathcal{N}(0, \sigma_j^2)$ for any $j \in \{g, a\}$, where ε_t^a denotes a productivity shock. The parameter $c_y = 1 - g_y - i_y$, with g_y being the share of exogenous government spending to output, whereas $i_y = (\gamma - 1 + \delta)k_y$ where γ is the steady-state growth rate, δ capital depreciation rate, k_y the steady-state capital to output ratio. Moreover, $z_y = r_k^* k_y$, where r_k^* is the steady-state rental rate of capital. The consumption Euler equation is described by

$$c_t = c_1 c_{t-1} + (1 - c_1)\mathbb{E}_t c_{t+1} + c_2(L_t - \mathbb{E}_t L_{t+1}) - c_3(r_t - \mathbb{E}_t \pi_{t+1}) + e_t^b \tag{C.2}$$

where L_t is supplied labor hours, e_t^b is a disturbance term, such that $e_t^b = \rho_b e_{t-1}^b + \varepsilon_t^b$ with $\varepsilon_t^b \sim \mathcal{N}(0, \sigma_b^2)$. Parameter $c_1 = (\lambda/\gamma)(1 + \lambda/\gamma)$, where λ denotes external consumption habit and σ_c the elasticity of intertemporal substitution. Moreover, $c_2 = (\sigma_c - 1)(w^* L^*/C^*)/(\sigma_c(1 + \lambda/\gamma))$ with $w^* L^*/C^*$ being the steady-state labor income share; and $c_3 = (1 - \lambda/\gamma)/(\sigma_c(1 + \lambda/\gamma))$. The equilibrium equation for investment, i_t, is

$$i_t = i_1 i_{t-1} + (1 - i_1)\mathbb{E}_t i_{t+1} + i_2 q_t + e_t^i \tag{C.3}$$

where q_t denotes the capital price, e_t^i is a disturbance to the investment-specific technology process, such that $e_t^i = \rho_i e_{t-1}^i + \varepsilon_t^i$ with $\varepsilon_t^i \sim \mathcal{N}(0, \sigma_i^2)$. Parameter $i_1 = (1 + \beta\gamma^{1-\sigma_c})^{-1}$, where β is the discount factor of households; $i_2 = \left((\gamma^2\varphi)(1 + \beta\gamma^{1-\sigma_c})\right)^{-1}$, with φ being the steady-state elasticity of the capital adjustment cost function. The equation for capital price is

$$q_t = q_1 \mathbb{E}_t i_{t+1} + (1 - q_1)\mathbb{E}_t r_{t+1}^k - (r_t - \mathbb{E}_t \pi_{t+1}) + q_2 e_t^b \tag{C.4}$$

where r_t is the nominal interest rate, r_t^k the rental rate of capital; $q_1 = \beta\gamma^{-\sigma_c}(1-\delta)$ and $q_2 = \sigma_c(\lambda+\gamma)/(\gamma-\lambda)$. The aggregate production function is

$$y_t = \phi_p(\alpha k_t^s + (1-\alpha)L_t + e_t^a) \tag{C.5}$$

where e_t^a is the TFP such that $e_t^a = \rho_a e_{t-1}^a + \varepsilon_t^a$ with $\varepsilon_t^a \sim \mathcal{N}(0, \sigma_a^2)$, α is the share of capital in production, and ϕ_p is the share of fixed costs in production plus unity.

$$k_t^s = k_{t-1} + z_t \tag{C.6}$$

where k_t^s denotes current capital used in production.

$$z_t = z_1 r_t^k \tag{C.7}$$

where $z_1 = (1-\psi)/\psi$ with ψ being a (positive) function of the elasticity of the capital utilization adjustment cost function. The equation for capital accumulation is described by

$$k_t = k_1 k_{t-1} + (1-k_1)i_t + k_2 e_t^i \tag{C.8}$$

where $k_1 = (1-\delta)/\gamma$; $k_2 = \gamma^2\varphi(1-k_1)(1+\beta\gamma^{1-\sigma_c})$. The equilibrium equation for the price mark-up follows

$$\mu_t^p = \alpha(k_t^s - L_t) + e_t^a - w_t \tag{C.9}$$

The Phillips equation is described by

$$\pi_t = \pi_1\pi_{t-1} + \pi_2\mathbb{E}_t\pi_{t+1} - \pi_3\mu_t^p + e_t^p \tag{C.10}$$

where e_t^p is a price mark-up disturbance assumed to follow an ARMA(1,1) process, $e_t^p = \rho_p e_{t-1}^p + \varepsilon_t^p - \mu_p\varepsilon_{t-1}^p$. Furthermore, $\pi_1 = \iota_p/(1+\beta\iota_p\gamma^{1-\sigma_c})$, with ι_p is the degree of indexation to past inflation; $\pi_2 = \beta\pi_1\gamma^{1-\sigma_c}/\iota_p$; $\pi_3 = \pi_1(1-\zeta_p)(1-\beta\zeta_p\gamma^{1-\sigma_c})/(\iota_p\zeta_p(1+\xi_p(\phi_p-1)))$. The rental rate of capital is given by

$$r_t^k = L_t - k_t + w_t \tag{C.11}$$

The equation for the wage mark-up is

$$\mu_t^w = w_t - \sigma_L L_t - \frac{\gamma c_t - \lambda c_{t-1}}{\gamma - \lambda} \tag{C.12}$$

where σ_L is the elasticity of labor supply with respect to the real wage. Real wages adjust according to

$$w_t = w_1 w_{t-1} + (1 - w_1)(\mathbb{E}_t w_{t+1} + \mathbb{E}_t \pi_{t+1}) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + e_t^w \tag{C.13}$$

where e_t^w is a disturbance to the wage mark-up following an ARMA(1,1) process, such that $e_t^w = \rho_w e_{t-1}^w + \varepsilon_t^w - \mu_w \varepsilon_{t-1}^w$. Moreover, $w_1 = 1/(1 + \beta\gamma^{1-\sigma_c})$; $w_2 = w_1(1 + \beta\iota_w\gamma^{1-\sigma_c})$ with ι_w being wage indexation; $w_3 = w_1\iota_w$; $w_4 = w_1(1 - \zeta_w)(1 - \beta\zeta_w\gamma^{1-\sigma_c})/(\zeta_w(1 + \xi_w(\phi_w - 1)))$, with ζ_w capturing real wage rigidity, ξ_w the curvature of the Kimball labor market aggregator, and $(\phi_w - 1)$ the steady-state labor market mark-up. Monetary policy sets the nominal interest rate according to the following Taylor rule:

$$r_t = \rho_{s_t} r_{t-1} + (1 - \rho_{s_t})(\phi_{s_t}^{\pi} \pi_t + \phi_{s_t}^{y}(y_t - y_{t-1})) + v_t \tag{C.14}$$

where $v_t = \rho_v v_{t-1} + \varepsilon_t^v$ with $\varepsilon_t^v \sim \mathcal{N}(0, \sigma_v^2)$ is a monetary shock.

The regime-dependent MSV solution of the model is given by

$$X_t = A_{s_t} X_{t-1} + B_{s_t} \epsilon_t \tag{C.15}$$

C.2 Estimation

For the estimation, we map a vector of observable variables, Y_t, with the endogenous variables vector X_t,

$$Y_t = \Psi_0 + \Psi_1 X_t \tag{C.16}$$

where Y_t contains data on output growth, consumption growth, investment growth, real wage growth, labor hours, inflation, and the federal funds rate.[35] Vector Ψ_0 is given by

$$\Psi_0 = \begin{bmatrix} \bar{\Delta y} & \bar{\Delta c} & \bar{\Delta i} & \bar{\Delta w} & \bar{l} & \bar{\pi} & \bar{r} \end{bmatrix}' \tag{C.17}$$

where $\bar{\Delta y}$, $\bar{\Delta c}$, $\bar{\Delta i}$, $\bar{\Delta w}$ are the average trend growth rates of output, consumption, investment, and real wage, respectively; $\bar{l}$ denotes steady state hours worked; $\bar{\pi}$ is the steady state inflation rate; and $\bar{r}$ is the steady state federal funds rate.

Tables 5 and ?? report the prior distribution characteristics as well as the estimated posterior mean, posterior mode, and the 5th and 95th percentiles of the posterior distribution for all 40

[35] We do not adjust the growth rates of output, consumption, and investment by population growth. Otherwise, we would have to make assumptions about the evolution of the population growth within the model since the SPF provides forecasts about output growth, not output growth per capita.

estimated parameters. As in Smets and Wouters (2007), we fix $\delta = 0.025$, $g_y = 0.18$, $\lambda_w = 1.5$, $\xi_w = 10$, and $\xi_p = 10$. Moreover, $\bar{r} = 100(\beta^{-1}\gamma^{\sigma_c}\pi^* - 1)$, where $\pi^* = 1 + \bar{\pi}/100$ and $\gamma = 1 + \bar{\Delta} y/100$.

XXXXX [36]

Table 5: Prior and posterior distribution of structural parameters

	Prior			Posterior			
	pdf	mean	std	mean	mode	5%	95%
Monetary policy parameters							
ϕ_1^{π}	$\mathcal{N}$	1.80	0.50	2.63	2.44	2.28	3.00
ϕ_2^{π}	$\mathcal{N}$	0.50	0.20	0.77	0.81	0.58	0.83
ϕ_1^{y}	$\mathcal{G}$	0.25	0.15	0.40	0.42	0.20	0.58
ϕ_2^{y}	$\mathcal{G}$	0.25	0.15	0.62	0.44	0.42	0.88
ρ_1	$\mathcal{B}$	0.60	0.20	0.64	0.61	0.57	0.75
ρ_2	$\mathcal{B}$	0.60	0.20	0.07	0.06	0.02	0.14
p_{12}	$\mathcal{B}$	0.0909	0.083	0.15	0.11	0.08	0.25
p_{21}	$\mathcal{B}$	0.0909	0.083	0.29	0.40	0.11	0.42
Other structural parameters							
α	$\mathcal{N}$	0.30	0.05	0.14	0.14	0.12	0.16
σ_c	$\mathcal{N}$	1.50	0.37	1.50	1.54	1.24	1.78
ϕ_p	$\mathcal{N}$	1.25	0.12	2.00	2.00	1.97	2.01
φ	$\mathcal{N}$	4	1.50	7.20	7.12	5.69	8.34
λ	$\mathcal{B}$	0.70	0.10	0.72	0.70	0.64	0.79
ζ_w	$\mathcal{B}$	0.50	0.10	0.71	0.70	0.63	0.78
σ_L	$\mathcal{N}$	2	0.75	1.99	2.02	1.66	2.38
ζ_p	$\mathcal{B}$	0.50	0.10	0.59	0.59	0.53	0.65
ι_w	$\mathcal{B}$	0.50	0.15	0.70	0.56	0.54	0.84
ι_p	$\mathcal{B}$	0.50	0.15	0.26	0.26	0.15	0.37
ψ	$\mathcal{B}$	0.50	0.15	0.43	0.44	0.32	0.56
μ_p	$\mathcal{B}$	0.50	0.20	0.65	0.64	0.53	0.75
μ_w	$\mathcal{B}$	0.50	0.20	0.80	0.79	0.69	0.88
$100(\beta^{-1} - 1)$	$\mathcal{G}$	0.25	0.10	0.14	0.15	0.08	0.21
$\bar{\Delta} y$	$\mathcal{N}$	0.40	0.10	0.21	0.22	0.19	0.23
$\bar{\pi}$	$\mathcal{G}$	0.62	0.10	0.58	0.63	0.49	0.67
$\bar{l}$	$\mathcal{N}$	0	2	-3.31	-2.12	-4.67	-2.03

Notes: The table reports the prior distribution characteristics for all the estimated parameters, excluding shocks' parameters. It then reports the estimated posterior mean and model, as well as the 5th and 95th percentiles of the posterior distributions, based on 500,000 draws generated through the Metropolis-Hastings algorithm.

In regime 1, the response of nominal interest rates to inflation deviations from the target is

[36] The one important difference relative to Bianchi (2013) is that we assume that the response of monetary policy to inflation in deviations from its target in the second regime is normally distributed with mean 0.5 and standard deviation 0.2. Differently, Bianchi (2013) assumes that that parameter has a gamma distribution with mean 1 and standard deviation 0.4. In our robustness exercises, we have found that the choice of prior does not affect the simulation results of the regime-robust FIRE test (details can be provided by the authors upon request).

normally distributed with mean 1.8 and standard deviation 0.5, whereas in regime 2 it is normally distributed with mean 0.5 and standard deviation 0.2. In both regimes, the response of nominal interest rates to output growth has a gamma distribution with mean 0.25 and standard deviation, whereas the persistence of nominal interest rates has a beta distribution with mean 0.6 and standard deviation 0.2. Both transition probabilities, p_{12} and p_{21}, are assumed to have a beta distribution with mean close to 0.1 and standard deviation 0.05.

The share of capital in production is normally distributed with mean 0.3 and standard deviation 0.05, whereas the share of fixed costs in production (plus unity) has a normal distribution centered at 1.25 with standard deviation 0.12. The elasticity of intertemporal substitution is normally distributed with mean 1.5 and standard deviation 0.37. The external consumption habit parameter has a beta distribution with mean 0.7 and standard deviation 0.1. The elasticity of labor supply with respect to the real wage is normally distributed with mean 2 and standard deviation 0.75. Wage and price indexation parameters both have a beta distribution with mean 0.5 and standard deviation 0.15, whereas real wage and price rigidity parameters have a beta distribution with mean 0.5 and standard deviation 0.1. The parameter linked to the elasticity of the capital utilization adjustment cost function, ψ, has a beta distribution with mean 0.5 and standard deviation 0.15.

The function of the households' discount factor, $100(\beta^{-1} - 1)$, is assumed to follow a gamma distribution with mean 0.25 and standard deviation 0.1. The average trend growth rate for output follows a normal distribution with mean 0.4 and standard deviation 0.1; steady-state inflation is assumed to follow a gamma distribution with mean 0.62 and standard deviation 0.1; hours worked in steady state are assumed to follow a normal distribution centered at 0 with standard deviation 2. Finally, the persistence of all the shocks has a beta prior with mean 0.5 and standard deviation 0.2. Furthermore, the prior of the standard deviation of each shock innovation is an inverse gamma distribution with mean 0.1 and standard deviation 2. The response of the price and wage mark-up disturbances to the past respective innovations in the ARMA(1,1) processed both follow a beta distribution with mean 0.5 and standard deviation 0.2. The response of the exogenous government spending to productivity innovations has beta distribution with mean 0.5 and standard deviation 0.2.

Table 6: Prior and posterior distribution of shock processes

	Prior			Posterior			
	pdf	mean	std	mean	mode	5%	95%
ρ_a	$\mathcal{B}$	0.5	0.2	0.36	0.36	0.34	0.38
ρ_b	$\mathcal{B}$	0.5	0.2	0.19	0.17	0.15	0.23
ρ_g	$\mathcal{B}$	0.5	0.2	0.45	0.45	0.42	0.48
ρ_i	$\mathcal{B}$	0.5	0.2	0.27	0.27	0.24	0.31
ρ_v	$\mathcal{B}$	0.5	0.2	0.16	0.16	0.15	0.18
ρ_p	$\mathcal{B}$	0.5	0.2	0.12	0.13	0.11	0.14
ρ_w	$\mathcal{B}$	0.5	0.2	0.38	0.38	0.35	0.41
ρ_{ga}	$\mathcal{B}$	0.50	0.20	0.66	0.67	0.55	0.77
μ_p	$\mathcal{B}$	0.50	0.20	0.65	0.64	0.53	0.75
μ_w	$\mathcal{B}$	0.50	0.20	0.80	0.79	0.69	0.88
σ_a	$\mathcal{IG}$	0.1	2	0.99	0.99	0.99	1.00
σ_b	$\mathcal{IG}$	0.1	2	0.39	0.46	0.27	0.52
σ_g	$\mathcal{IG}$	0.1	2	0.93	0.94	0.88	0.96
σ_i	$\mathcal{IG}$	0.1	2	0.81	0.82	0.73	0.88
σ_v	$\mathcal{IG}$	0.1	2	0.49	0.41	0.38	0.59
σ_p	$\mathcal{IG}$	0.1	2	0.87	0.86	0.83	0.91
σ_w	$\mathcal{IG}$	0.1	2	0.86	0.85	0.75	0.92

Notes: The table reports the prior distribution characteristics for all the estimated parameters describing shock processes. It then reports the estimated posterior mean and model, as well as the 5th and 95th percentiles of the posterior distributions, based on 500,000 draws generated through the Metropolis-Hastings algorithm.

Turning to the posterior distribution, we plot the kernel densities in Figure 8.

C.3 Filtering and smoothing algorithms

In what follows, we describe the Kim and Nelson (1999) filtering and smoothing algorithms, given the state-space representation of the model in (C.15)-(C.16) in the main text. We initiate the filtering process at regime $s_0 = 1$, thus $Pr(s_0) = \frac{1-p_{22}}{2-p_{11}-p_{22}}$. Moreover, $X^{s_0}_{0|0} = \mathbf{0}_{n_x \times 1}$ and $vec(K^{s_0}_{0|0}) = \left(I_{n_x^2} - (A_{s_0} \otimes A_{s_0})\right)^{-1} (B_{s_0} \otimes B_{s_0}) vec(\Sigma)$. Then, for any $t \geq 1$, we abide by the following filtering algorithm:

1. Kalman filter

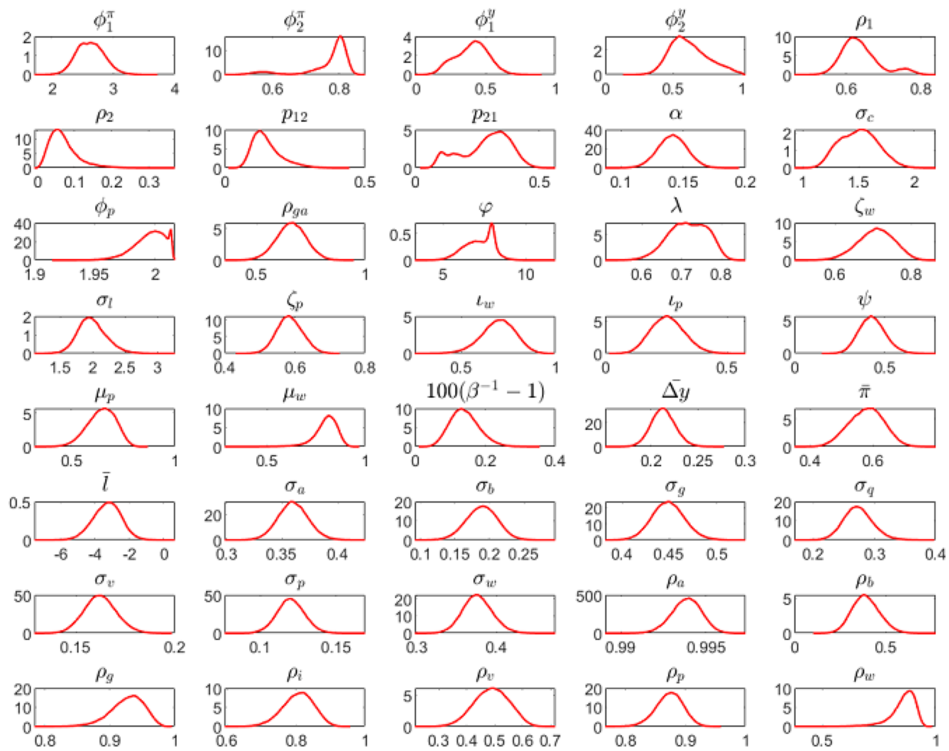
$$X^{(s_{t-1},s_t)}_{t|t-1} = A_{s_t} X^{s_t}_{t-1|t-1} \tag{C.18}$$

$$K^{(s_{t-1},s_t)}_{t|t-1} = A_{s_t} K^{s_t}_{t-1|t-1} A'_{s_t} + B_{s_t} \Sigma B'_{s_t} \tag{C.19}$$

$$g^{(s_{t-1},s_t)}_{t|t-1} = Y_t - \Psi_1 X^{(s_{t-1},s_t)}_{t|t-1} - \Psi_0 \tag{C.20}$$

$$X^{(s_{t-1},s_t)}_{t|t} = X^{(s_{t-1},s_t)}_{t|t} + K^{(s_{t-1},s_t)}_{t|t-1} \Psi'_1 \left(\Psi_1 K^{(s_{t-1},s_t)}_{t|t-1} \Psi'_1\right)^{-1} g^{(s_{t-1},s_t)}_{t|t-1} \tag{C.21}$$

Figure 8: Posterior distribution of all the estimated parameters



Notes: The figure exhibits the kernel density of the posterior distribution of all the estimated parameters, based on 500,000 draws generated through the Metropolis-Hastings algorithm.

$$K_{t|t}^{(s_{t-1},s_t)} = \left(I - K_{t|t-1}^{(s_{t-1},s_t)} \Psi_1' \left(\Psi_1 K_{t|t-1}^{(s_{t-1},s_t)} \Psi_1' \right)^{-1} \Psi_1 \right) K_{t|t-1}^{(s_{t-1},s_t)} \tag{C.22}$$

2. Hamilton filter

   Let $\mathcal{I}_t$ denote the information set up until period t.

$$Pr(s_t, s_{t-1}|\mathcal{I}_{t-1}) = Pr(s_t|s_{t-1})Pr(s_{t-1}|\mathcal{I}_{t-1}) \tag{C.23}$$

$$f(Y_t|\mathcal{I}_{t-1}) = \sum_{s_t} \sum_{s_{t-1}} f(Y_t|s_t, s_{t-1}, \mathcal{I}_{t-1}) Pr(s_t, s_{t-1}|\mathcal{I}_{t-1}) \tag{C.24}$$

   where

$$f(Y_t|s_t, s_{t-1}, \mathcal{I}_{t-1}) = (2\pi)^{-\frac{ny}{2}} \mid \Psi_1 K_{t|t-1}^{(s_{t-1},s_t)} \Psi_1' \mid^{-\frac{1}{2}} exp \left(-\frac{1}{2} g_{t|t-1}^{(s_{t-1},s_t)'} \left(\Psi_1 K_{t|t-1}^{(s_{t-1},s_t)} \Psi_1' \right)^{-1} g_{t|t-1}^{(s_{t-1},s_t)} \right) \tag{C.25}$$

$$Pr(s_t, s_{t-1}|\mathcal{I}_t) = \frac{f(Y_t|s_t, s_{t-1}, \mathcal{I}_{t-1})Pr(s_t, s_{t-1}|\mathcal{I}_{t-1})}{f(Y_t|\mathcal{I}_{t-1})} \tag{C.26}$$

$$Pr(s_t|\mathcal{I}_t) = \sum_{s_{t-1}} Pr(s_t, s_{t-1}|\mathcal{I}_t) \tag{C.27}$$

3. Approximations

$$X_{t|t}^{s_t} = \frac{\sum_{s_{t-1}} Pr(s_t, s_{t-1}|\mathcal{I}_t) X_{t|t}^{(s_{t-1}, s_t)}}{Pr(s_t|\mathcal{I}_t)} \tag{C.28}$$

$$K_{t|t}^{s_t} = \frac{\sum_{s_{t-1}} Pr(s_t, s_{t-1}|\mathcal{I}_t) \left(K_{t|t}^{(s_{t-1}, s_t)} + (X_{t|t}^{s_t} - X_{t|t}^{(s_{t-1}, s_t)})(X_{t|t}^{s_t} - X_{t|t}^{(s_{t-1}, s_t)})'\right)}{Pr(s_t|\mathcal{I}_t)} \tag{C.29}$$

We now turn to the smoothing algorithm. We are particularly interested on the evolution of the smoothed regime probabilities that will help us make inferences about the regime path, and the evolution of smoothed $X_{t|T}^{s_t}$ for each regime s_t, where T denotes the final period of the sample. Starting from $t+1 = T$, we have

$$Pr(s_t, s_{t+1}|\mathcal{I}_T) = \frac{Pr(s_{t+1}|\mathcal{I}_T)Pr(s_t|\mathcal{I}_t)Pr(s_{t+1}|s_t)}{Pr(s_{t+1}|\mathcal{I}_t)} \tag{C.30}$$

where $Pr(s_{t+1}|\mathcal{I}_t) = Pr(s_{t+1}|s_t)Pr(s_t|\mathcal{I}_t)$. Finally, the smoothed regime probabilities are given by

$$Pr(s_t|\mathcal{I}_T) = \sum_{s_{t+1}} Pr(s_t, s_{t+1}|\mathcal{I}_T) \tag{C.31}$$

Regarding the smoothing algorithm for X_t, we first compute

$$X_{t|T}^{(s_t, s_{t+1})} = X_{t|t}^{s_t} + \widetilde{K}_t^{(s_t, s_{t+1})}(X_{t+1|T}^{s_{t+1}} - X_{t+1|t}^{(s_t, s_{t+1})}) \tag{C.32}$$

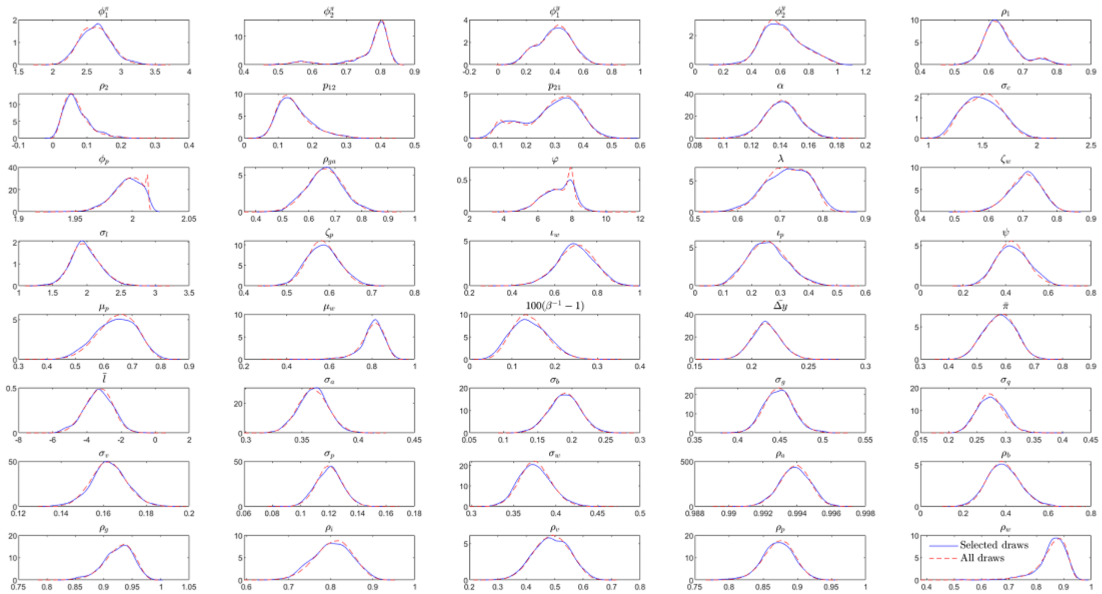
where $\widetilde{K}_t^{(s_t, s_{t+1})} = K_{t|t}^{s_t} A'_{s_{t+1}} \left(K_{t+1|t}^{(s_t, s_{t+1})}\right)^{-1}$. Further,

$$K_{t|T}^{(s_t, s_{t+1})} = K_{t|t}^{s_t} + \widetilde{K}_t^{(s_t, s_{t+1})}(K_{t+1|T}^{s_{t+1}} - K_{t+1|t}^{(s_t, s_{t+1})}) \left(\widetilde{K}_t^{(s_t, s_{t+1})}\right)' \tag{C.33}$$

$$X_{t|T}^{s_t} = \frac{\sum_{s_{t+1}} Pr(s_t, s_{t=1}|\mathcal{I}_T) X_{t|T}^{(s_t, s_{t+1})}}{Pr(s_t|\mathcal{I}_T)} \tag{C.34}$$

$$K_{t|T}^{s_t} = \frac{\sum_{s_{t+1}} Pr(s_t, s_{t+1}|\mathcal{I}_T) \left(K_{t|T}^{(s_t, s_{t+1})} + (X_{t|T}^{s_t} - X_{t|T}^{(s_t, s_{t+1})})(X_{t|T}^{s_t} - X_{t|T}^{(s_t, s_{t+1})})'\right)}{Pr(s_t|\mathcal{I}_T)} \tag{C.35}$$

Figure 9: Posterior distribution



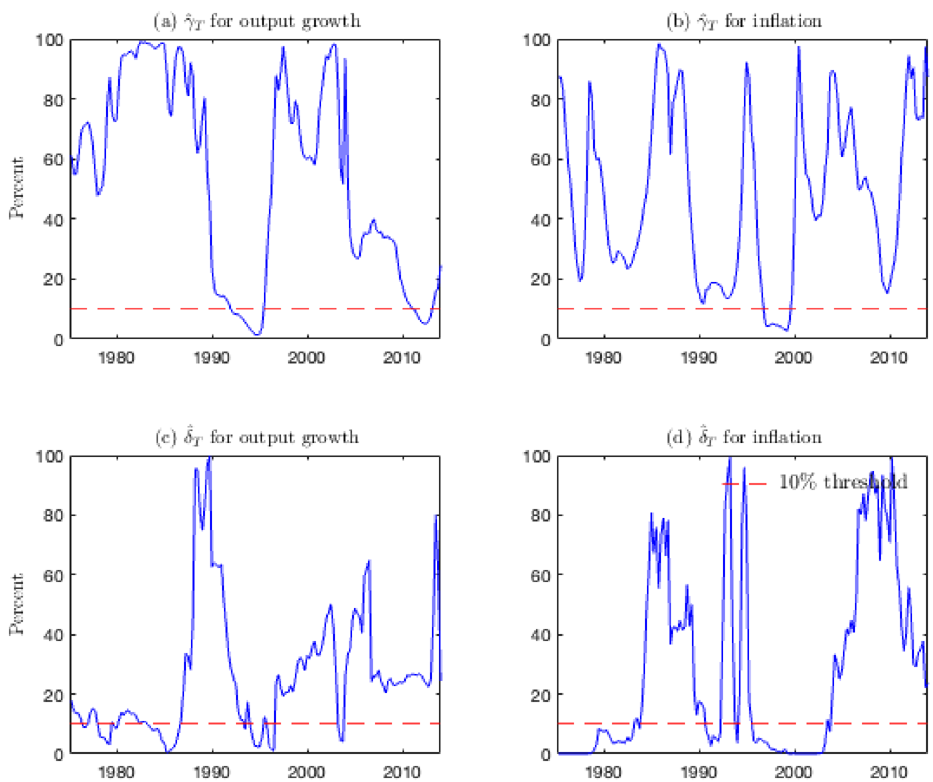
Notes: The figure plots the kernel densities of the full posterior distribution (500,000 draws) in dashed red jointly with the respective densities of the 1,000 draws used for the two testing approaches in solid blue.

C.4 Regime-robust FIRE test: additional results

Figure plots the kernel density of the full posterior distribution of 500,000 draws jointly with the kernel density of the $N = 1,000$ draws used for the two test approaches. As the figure shows, the 1,000 draws represent well the full posterior distribution of the estimated parameters.

Figure 10 plots the evolution of p-values associated with the FIRE test for waves of over- and under-reaction in the SPF data. We note that subsamples when the SPF estimate is significantly different from what is implied by FIRE at 90% confidence level, that is, when the whole confidence interval falls in a strictly positive or negative region in Figure 7, coincide with subsamples for which the p-value is less than 10% in Figure 10.

Figure 10: FIRE test for waves of over- and under-reaction: p-values



Notes: The figure shows p-values of the null that the that the empirical estimates of $\hat{\gamma}_t$ and $\hat{\delta}_t$ in (1) and (2) were generated by the DSGE-RS model under FIRE for each 40-quarter rolling window. The values are centered at the midpoint of the rolling regression window (e.g. 1980 denotes the regression window 1975:1 to 1984:4). The dashed red line indicates the 10% significance level for rejection of the null.