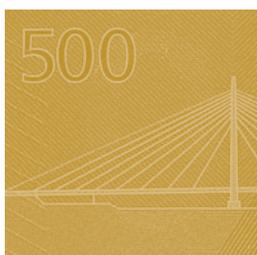




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### A THEORY OF PRUNING

Giovanni Lombardo and Harald Uhlig



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## **Abstract**

Often, numerical simulations for dynamic, stochastic models in economics are needed. Higher order methods can be attractive, but bear the danger of generating explosive solutions in originally stationary models. Kim-Kim-Schaumburg-Sims (2008) proposed pruning to deal with this challenge for second order approximations. In this paper, we provide a theory of pruning and formulas for pruning of any order. We relate it to results described by Judd (1998) on perturbing dynamical systems.

**Keywords:** pruning, numerical simulation, numerical economics, Taylor expansion, perturbation methods

**JEL codes:** C63, C02, C62

## Non-technical summary

Solving exactly non-linear, dynamic, stochastic models for policy analysis is currently not feasible, except for particularly simple models. Perturbation methods have successfully been used in physics as well as in economics to circumvent this problem and obtain approximate solutions. The seminal work of Judd (1998) has generated considerable research directed at adapting this method to economic problems. An apparent drawback of the proposed perturbation technique is that it is not well suited for simulations and computation of model-based statistics of interest. This is because the solution consists of a non-linear state-space representation, which away from the point of approximation, can generate non-stationary solutions, which in principle could have nothing to do with the dynamic properties of the original problem.

In order to avoid this problem, Kim et al. (2008) have proposed a practical correction of the non-linear state-space representation that guarantees stability of approximations up to second order of accuracy. This technique consists of “pruning” the terms of the solution that can generate instability.

A number of papers have recently attempted to provide theoretical foundations to the pruning approach. Our paper shows that if we represent the dynamical system in terms of the perturbation parameter (typically the standard deviation of the shocks) and of variables invariant to this parameter, a perturbation solution exists which has the property of being stationary as long as the original system is stationary around the point of approximation. We show that this perturbation approach is one of the approaches discussed in Judd’s (1998) textbook (Chapter 13), as well as in a number of mathematical articles, and that this approach generates solutions that are identical to those obtained under pruning. In this sense we provide a theory of pruning.

We derive general formulas for the computation of approximate solutions to any order of accuracy. Hence, we believe that our contribution can offer

a theory-based guide on higher order approximations of models useful for policy analysis.

# 1 Introduction

*When both the true nonlinear model and the second-order approximate model are stationary and ergodic, and the true unconditional expectation in question is a twice differentiable function of  $\sigma$  in the neighborhood of  $\sigma = 0$ , then it is possible to estimate the expectation from long simulations of the approximate model, with the estimates accurate locally in  $\sigma$  in the usual sense ( quoted from Kim et al. (2008), p. 3410).*

Solutions to dynamic stochastic equilibrium (DSGE) models can often be thought of to take a recursive form, say

$$x_t = g(x_{t-1}; \sigma) + \sigma \epsilon_t \quad (1)$$

In this equation,  $x_t$  denotes the endogenous variables,  $\epsilon_t$  a disturbance with some given distribution, and  $\sigma$  controls the variance of  $\epsilon_t$ . The law of motion  $g(\cdot)$  is not known: rather, it needs to be derived from the underlying equations of the DSGE model. The same holds true for statistics of interests regarding the resulting time series  $x_t$ , such as moments which may be suitable for GMM estimation.

Indeed, it is typically not possible to provide an exact or closed-form expression for  $g(\cdot)$  or these statistics. Instead, numerical approximation methods need to be used, see Judd (1998). First-order approximations are well-understood and popular, see e.g. Uhlig (1999). Following Jin and Judd (2002), researchers have made increasing use of higher-order polynomial approximations to  $g$ , obtained from a perturbation approach or Taylor series around some steady state. These higher-order approximations contain quadratic terms in  $x_{t-1}$ . As Kim et al. (2008), henceforth KKSS, have pointed out, these higher order terms can be problematic for simulations. In essence, the quadratic terms generate an unstable dynamic, if the argument

$x_{t-1}$  is pushed sufficiently far away from the steady state, and it may be hard or not practical to rule out such behavior. KKSS suggest “pruning” to deal with this problem. Briefly put, higher-order terms get rewritten, per relating them to lower- order terms, in order to render the simulations stationary. While pruning has become popular, questions linger. One may wonder whether there is a deeper rationale for this method or whether it just works by “miracle” of good judgement. There is a debate in the literature on how to generalize pruning to third and higher order, as well as to treat constants. Furthermore, the method has come under some attack (see den Haan and de Wind, 2012).

The literature has largely proceeded by proposing specific solutions and checking numerical adequacy for specific examples. We seek to complement this literature by, in essence, providing a theory of pruning. We show that pruning can be understood as a standard Taylor approximation, when stating the variable of interest at date  $t$  as a function of the standard deviation parameter  $\sigma$  as well as variables which are invariant to it. We show that it is tightly related to an approach provided by Judd (1998) on perturbing dynamical systems. A key contribution of this paper is to solidify the appropriate third-order pruning scheme, to provide the forth-order pruning scheme, and to provide an algorithm for computing schemes of any order.

The paper proceeds as follows. In section 2, we relate our contribution to the existing literature. In section 3, we provide a short introduction to DSGE models, their solutions and pruning. This will help to fix notation. It leads into section 4, where we show that pruning can be understood as a standard Taylor approximation, but for the “right” function. Section 5 contains our key results and provides the formulas for pruning of any order. We explicitly state the formulas for pruning of third and forth order. Section 6 shows that pruning, in essence, has already been obtained in Judd (1998).

## 2 Related literature

Following Judd's (1998) seminal contribution on perturbation methods in economics, a number of papers have discussed various aspects of this solution technique. Jin and Judd (2002) provide an earlier guide on the application of perturbation methods to DSGE models, while the often cited work by Schmitt-Grohé and Uribe (2004) further clarifies the technique, provides computer codes to implement it, and discusses the role of risk in second-order approximations.

One practical issue that remained unresolved in the literature consisted in dealing with the inherent non-linearity of the state-space representation of the higher-order solution obtained following these techniques.<sup>1</sup> Lombardo and Sutherland (2007) discuss an approach based on "order matching" for second order approximations that generates a recursively-linear state-space representation and, hence, avoids the problem of "spurious" explosive paths in simulations. Their approach shares many formal similarities with the method discussed here. Nevertheless, and contrary to our paper, they don't attempt to provide mathematical foundations to their approach, nor do they link it to "pruning".

KKSS address explicitly the problem of the non-linearity of the state-space representation of a second-order perturbation-based solution. After discussing the limitation of the prevalent perturbation approach (i.e. à la Jin and Judd, 2002) they propose a practical solution consisting of "pruning" the state-space representation, and thus turning it into a recursively-linear system of equations. The details of this approach are discussed further below.

More recently, a number of papers have investigated the properties as well as the foundations of the "pruning" technique. In particular den Haan

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<sup>1</sup>While Schmitt-Grohé and Uribe (2004) don't discuss this problem, their computer codes apply "pruning" to second order.

and de Wind (2012) emphasize the potential pitfalls of particular implementations of the “pruning” technique to higher orders of approximation, and propose an alternative approach. Andreasen et al. (2013) extend the “pruning” technique to higher order of approximation using an “order-matching” argument. While their approach generates solutions that are formally similar to ours, they don’t discuss the link between the proposed technique and the perturbation approach studied in this paper and discussed in the mathematical literature (e.g. Holmes, 1995 and Murdock, 1987) and in Judd (Ch. 13 1998).<sup>2</sup>

An alternative, interesting approach is followed by Lan and Meyer-Gohde (2013). Similarly to us, they show that pruning can be understood as a Taylor expansion for a function in the appropriate domain. Contrary to us, they base their approximation on an infinite moving average representation.

Finally our work is deeply linked to the exposition of the perturbation approach provided by Kenneth Judd in Chapter 13 of his textbook (Judd, 1998). In his discussion of perturbation methods (e.g. see Section 13.3), Judd refers to an approach widely studied in the mathematical literature (e.g. Holmes, 1995 and Murdock, 1987) which we think provides the theoretical foundations of KKSS’ “pruning” technique. Given the centrality of Judd’s contribution to our analysis, we provide a more thorough comparison in Section 6.

### **3 DSGE models and pruning**

The purpose of this section is to provide a quick introduction into pruning and the key issues to set the stage for section 4. In order to keep notation

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<sup>2</sup>Lombardo (2010), discusses how to follow this mathematical literature to obtain recursively-linear state-space representations. Our paper replaces and extends his earlier contribution.

palatable, we will assume for now, that  $x_t$  is real-valued, that  $\bar{x}$  is the non-stochastic steady state, etc.. We provide results for the vector-valued case in section 5.

We suppose that the equations characterizing the dynamics of an economic model can be written as

$$0 = E_t [H(x_{t-1}, x_t, x_{t+1}, \sigma\epsilon_t; \sigma)], t \geq 1 \quad (2)$$

where  $x_t \in \mathbf{R}$  is an endogenous variable with a given initial value  $x_0$ ,  $\epsilon_t$  is an iid sequence of random variables  $\epsilon_t \sim F$  for some given distribution  $F$  and with  $E_{t-1}[\epsilon_t] = 0$ ,  $\sigma \geq 0$  is a parameter and  $E_t$  is the expectation with respect to the information given by  $(x_0, \dots, x_t, \epsilon_1, \dots, \epsilon_t)$ . The function  $H(\cdot)$  is a primitive of the model, and may depend on additional parameters.

We shall constrain our attention to the situation, where  $\epsilon_t$  is the only stochastic driving force<sup>3</sup> and therefore constrain  $x_t$  to be measurable with respect to the  $\sigma$ -algebra or information  $\mathcal{F}_t$  generated by  $\epsilon_1, \dots, \epsilon_t$ . A solution is a  $\mathcal{F}_t$ -measurable stochastic sequence  $x_t$ , satisfying (2). Conceptually, a solution can be written as

$$x_t = f_t(\epsilon_1, \dots, \epsilon_t; x_0, \sigma) \quad (3)$$

Obviously, obtaining the unknown function  $f(\cdot)$  by analyzing (2) is typically far from trivial, and we will not assume that  $f(\cdot)$  is known. Often, one can show that the solution takes a recursive form,

$$x_t = g(x_{t-1}, \epsilon_t; \sigma) \quad (4)$$

or even

$$x_t = g(x_{t-1}; \sigma) + \sigma\epsilon_t \quad (5)$$

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<sup>3</sup>Note that this implies that sunspot solutions must be functions of the exogenous sequence  $\epsilon_t$ .

We shall assume that the second case (5) applies. Many researchers find it convenient to work with the recursive form rather than (3). Once again, obtaining the unknown function  $g(\cdot)$  by analyzing (2) is typically far from trivial, and we will not assume that  $g(\cdot)$  is known. Often, however, it is possible to obtain usefully accurate values for  $g$  and several orders of its derivatives at some point  $\bar{x}$ , see Judd (1998). For example,  $\bar{x}$  may be the non-stochastic steady state of (2) and thus the value which satisfies

$$\bar{x} = g(\bar{x}; 0) \tag{6}$$

With this, one can obtain a Taylor expansion of  $g$  around  $(\bar{x}; 0)$ , up to some order. A second order Taylor expansion delivers

$$\begin{aligned} x_t = & g(\bar{x}; 0) + g_x(\bar{x}; 0)(x_{t-1} - \bar{x}) + g_\sigma(\bar{x}; 0)\sigma \\ & + \frac{1}{2} (g_{xx}(\bar{x}; 0)(x_{t-1} - \bar{x})^2 + g_{x\sigma}(\bar{x}; 0)(x_{t-1} - \bar{x})\sigma + g_{\sigma\sigma}(\bar{x}; 0)\sigma^2) \\ & + \sigma\epsilon_t + O(|x_{t-1} - \bar{x}|^3, \sigma^3) \end{aligned}$$

where  $O(|x_{t-1} - \bar{x}|^3, \sigma^3)$  denotes a term that is a bounded function of  $|x_{t-1} - \bar{x}|^3$  and  $\sigma^3$  and therefore vanishes near  $(\bar{x}, 0)$ . For simplicity of exposition, let us assume (6) and furthermore, that  $\bar{x} = 0$ . Schmitt-Grohé and Uribe (2004) have shown that  $g_\sigma(\bar{x}; 0) = 0$  and  $g_{x\sigma}(\bar{x}; 0) = 0$  at the nonstochastic steady state  $\bar{x}$ . Dropping the argument  $(\bar{x}; 0)$ , the equation above can then be written as

$$x_t \approx g_x x_{t-1} + \frac{1}{2} (g_{xx} x_{t-1}^2 + g_{\sigma\sigma} \sigma^2) + \sigma\epsilon_t \tag{7}$$

where “ $\approx$ ” here means up to a function  $O(|x_{t-1}|^3, \sigma^3)$ .

As KKSS (2008) have pointed out, iteration on this equation can be explosive, even with  $|g_x| < 1$ , due to the presence of the quadratic term  $g_{xx} x_{t-1}^2$ . This can lead to problems, when simulating long samples from (7). Their suggestion is to “prune” (7) for the purpose of simulations, and to instead use

$$x_t \approx \tilde{x}_t^{(2)} \tag{8}$$

where

$$\tilde{x}_t^{(1)} = g_x \tilde{x}_{t-1}^{(1)} + \sigma \epsilon_t \quad (9)$$

$$\tilde{x}_t^{(2)} = g_x \tilde{x}_{t-1}^{(2)} + \frac{1}{2} \left( g_{xx} \tilde{x}_{t-1}^{(1)} \tilde{x}_{t-1}^{(1)} + g_{\sigma\sigma} \sigma^2 \right) + \sigma \epsilon_t \quad (10)$$

It is straightforward to see that  $x_t^{(2)}$  is stationary, provided that  $|g_x| < 1$ . The pruning formulas moreover share similarities with the quadratic approximation (7): the formula for  $x_t^{(2)}$  is the same as in (7), except that the higher-order terms are not in terms of its own past, but are in terms of the past of  $x_t^{(1)}$  instead. While these properties are attractive, and while these formulas have a good intuitive appeal, it is perhaps less clear on more formal grounds, why  $x_t^{(2)}$  is close to the true stochastic process of the model. Moreover, while the formulas are suggestive regarding their generalizations to higher orders, one may debate exactly on how to proceed. Indeed, as pointed out in the introduction, there is considerable debate in the literature already.

## 4 A different Taylor expansion

In order to provide a theory of pruning, we return to equation (3), i.e.

$$x_t = f_t(\epsilon_1, \dots, \epsilon_t; x_0, \sigma)$$

While we seek to calculate good approximations for  $x_t$  at some given initial condition  $x_0$  as well as some target value for  $\sigma$ , say,  $\sigma^*$ , there is some liberty as to what initial condition one ought to assume at  $\sigma = 0$ , the point of approximation. It turns out that the following assumption is particularly convenient.

**Assumption A. 1** *The series  $x_t = f_t(\epsilon_1, \dots, \epsilon_t; x_0, \sigma)$  solves problem (2) for the sequence of shocks  $\epsilon_1, \epsilon_2, \dots, \sigma$  and the the initial condition*

$$x_0(\sigma) = \bar{x} + \frac{\sigma}{\sigma^*} (x_0 - \bar{x}) \quad (11)$$

where  $\bar{x}$  is the non-stochastic steady state.

As a result,  $\bar{x}$  is the initial condition at  $\sigma = 0$  and  $x_0$  is the initial condition at  $\sigma = \sigma^*$ . Moreover,

$$\frac{\partial f_0(x_0, \sigma)}{\partial \sigma} = \frac{\partial x_0(\sigma)}{\partial \sigma} = \frac{1}{\sigma^*} (x_0 - \bar{x}) \quad (12)$$

We are now interested in characterizing the Taylor expansion for  $f_t$  in terms of  $\sigma$ . We shall do so by exploiting the properties of the recursive law of motion (5). We will show that one obtains pruning naturally. For this, perhaps the most important property of (3) can be stated as follows. Rewrite that equation as

$$x_t = f_t(\theta_t; \sigma) \quad (13)$$

where

$$\theta_t = (x_0, \epsilon_1, \dots, \epsilon_t) \quad (14)$$

Note that  $\theta_t$  is invariant with respect to  $\sigma$ . Therefore, when taking derivatives with respect to  $\sigma$ , one does not have to “worry” about the endogenous impact on  $\theta_t$ . Compare (13) with (4): while formally similar, the argument  $x_{t-1}$  will move with  $\sigma$ . It is for that reason that we need to start the analysis here with (3) rather than (4).

We shall focus first on the case of a second-order Taylor expansion for  $f_t$ . Note that

$$\bar{x} = f_t(\theta_t; 0) \quad (15)$$

For notational simplicity, assume that

$$\bar{x} = 0. \quad (16)$$

Let

$$x_t^{(j)} = \frac{\partial^j f_t}{\partial \sigma^j} \Big|_{(\theta_t; \bar{x}, 0)}$$

For a given  $\theta_t$ , the second-order Taylor expansion of  $f_t$  with respect to  $\sigma$  is

$$x_t \approx \bar{x} + x_t^{(1)}\sigma + \frac{1}{2}x_t^{(2)}(\sigma)^2 \quad (17)$$

where “ $\approx$ ” means up to a function  $O(\sigma^3)$ . For example, note that (12) implies that

$$x_0^{(1)} = \frac{1}{\sigma}(x_0 - \bar{x}) \quad (18)$$

$$x_0^{(j)} = 0, \text{ for } j \geq 2 \quad (19)$$

We obtain the following result. It is a simple consequence of (5), the chain rule as well as  $g_\sigma(\bar{x}; 0) = 0$  and  $g_{x\sigma}(\bar{x}; 0) = 0$ .

**Proposition 1**

$$x_t^{(1)} = g_x x_{t-1}^{(1)} + \epsilon_t \quad (20)$$

$$x_t^{(2)} = g_x x_{t-1}^{(2)} + g_{xx} \left( x_{t-1}^{(1)} \right)^2 + g_{\sigma\sigma} \quad (21)$$

**Proof:** *The notation in the proof may look daunting, but it just writing out the simple consequences of (5), the chain rule as well as  $g_\sigma(\bar{x}; 0) = 0$  and  $g_{x\sigma}(\bar{x}; 0) = 0$ , proceeding by induction. For a given  $\theta_t$  and some arbitrary  $\sigma$ , exploiting (5) and the chain rule delivers*

$$\frac{\partial f_t}{\partial \sigma}(\theta_t; \bar{x}, \sigma) = \frac{\partial}{\partial \sigma} (g(f_{t-1}(\theta_{t-1}; \bar{x}, \sigma); \sigma) + \sigma \epsilon_t) \quad (22)$$

$$= g_x(f_{t-1}(\theta_t; \bar{x}, \sigma); \sigma) \frac{\partial f_{t-1}}{\partial \sigma}(\theta_t; \bar{x}, \sigma) + g_\sigma(f_{t-1}(\theta_t; \bar{x}, \sigma); \sigma) + \epsilon_t$$

for the first derivative with respect to  $\sigma$  and, taking the derivative again with respect to  $\sigma$ ,

$$\begin{aligned} \frac{\partial^2 f_t}{\partial \sigma^2}(\theta_t; \bar{x}, \sigma) &= \frac{\partial}{\partial \sigma} \left( g_x(f_{t-1}(\theta_t; \bar{x}, \sigma); \sigma) \frac{\partial f_{t-1}}{\partial \sigma}(\theta_t; \bar{x}, \sigma) + g_\sigma(f_{t-1}(\theta_t; \bar{x}, \sigma); \sigma) + \epsilon_t \right) \\ &= g_x(f_{t-1}(\theta_t; \bar{x}, \sigma); \sigma) \frac{\partial^2 f_{t-1}}{\partial \sigma^2}(\theta_t; \bar{x}, \sigma) \\ &\quad + g_{xx}(f_{t-1}(\theta_t; \bar{x}, \sigma); \sigma) \left( \frac{\partial f_{t-1}}{\partial \sigma}(\theta_t; \bar{x}, \sigma) \right)^2 \\ &\quad + 2g_{x\sigma}(f_{t-1}(\theta_t; \bar{x}, \sigma); \sigma) \frac{\partial f_{t-1}}{\partial \sigma}(\theta_t; \bar{x}, \sigma) + g_{\sigma\sigma}(f_{t-1}(\theta_t; \bar{x}, \sigma); \sigma) \end{aligned} \quad (23)$$

Evaluate these expressions at  $\sigma = 0$  and exploit  $g_\sigma(\bar{x}; 0) = 0$  and  $g_{x\sigma}(\bar{x}; 0) = 0$  to obtain (20) and (21) •

Equations (20) and (21) may already look rather similar to (9) and (10). To make the comparison easier, we introduce the additional notation

$$\hat{x}_t^{(j)} = x_t^{(j)} \sigma^j / j! \quad (24)$$

We obtain

$$\hat{x}_t^{(1)} = g_x(\cdot) \hat{x}_{t-1}^{(1)} + \sigma \epsilon_t \quad (25)$$

$$\hat{x}_t^{(2)} = g_x(\cdot) \hat{x}_{t-1}^{(2)} + \frac{1}{2} \left( g_{xx}(\cdot) \hat{x}_{t-1}^{(1)} \hat{x}_{t-1}^{(1)} + g_{\sigma\sigma}(\cdot) \sigma^2 \right) \quad (26)$$

We approximate

$$x_t \approx \hat{x}_t^{(1)} + \hat{x}_t^{(2)}$$

per (17). Already,  $\hat{x}_t^{(1)}$  here is equal to  $x_t^{(1)}$  in (9). It remains to show that  $\hat{x}_t^{(1)} + \hat{x}_t^{(2)}$  here equals  $x_t^{(2)}$  in (10), provided this is true initially. But this follows now easily by complete induction.

**Proposition 2** *Suppose that*

$$\hat{x}_t^{(1)} = x_t^{(1)} \quad (27)$$

for  $t = 0$  and that

$$\hat{x}_t^{(1)} + \hat{x}_t^{(2)} = x_t^{(2)} \quad (28)$$

for  $t = 0$ . Then (27) and (28) hold true for all  $t \geq 0$ .

**Proof:** This is true for  $t = 0$  by assumption. Suppose this is true for both equations for  $t - 1$ . It is then trivial to see that (27) is true for  $t$  and it is a simple calculation to see that it is true for (28) as well. In words: the second-order Taylor expansion here coincides with the pruned solution

of KKSS, when using the same initial conditions. •

We therefore obtain pruning as an exact Taylor expansion of second order to  $f(\theta_t; \sigma)$  in  $\sigma$  at  $\sigma = 0$ . This is a key insight of this paper. It may be worth noting that the starting point for the pruning simulation here is

$$\begin{aligned}\hat{x}_0^{(1)} &= x_0 \\ \hat{x}_0^{(2)} &= 0\end{aligned}$$

as a consequence of (18) and (24).

## 5 Higher-order pruning

In this section we derive a general formula for the  $x_t^{(m)}$  terms, with  $m \in \mathbb{N} \cup \{0\}$ . We follow the strategy of the proof for proposition 1, but expanding the Taylor series to length  $m$  rather than 2. More specifically, we compare the coefficients of the  $m$ -th order Taylor expansion of  $x_t = f_t(\sigma)$  to the chain-rule induced Taylor expansion of  $x_t = g(f_{t-1}(\sigma); \sigma) + \sigma \epsilon_t$ . To do the latter, we proceed in three steps:

1. take the  $n$ -th order Taylor expansion of  $g(x_{t-1}, \sigma)$  in  $x_{t-1}$  and  $\sigma$  around  $\bar{x}$  and 0, respectively;
2. replace the terms in  $(x_{t-1} - \bar{x})^j$ ,  $j = 0, \dots, n$ , with the  $n$ -th order Taylor expansion of  $f_{t-1}$  in  $\sigma$ ,
3. match coefficients in  $\sigma$  to obtain the terms of interest.

For the first two steps, we can draw on the well-known expressions for bivariate Taylor expansions and multinomial formulas.

**Proposition 3** *Suppose that  $\bar{x} = 0$ , wlog. Then,*

$$x_t \approx \sum_{i=1}^m \hat{x}_t^{(i)} \quad (29)$$

where  $\hat{x}_t^{(i)}$  satisfy the recursion

$$\hat{x}_t^{(i)} = 1_{i=1} \sigma \epsilon_t + \sum_{(k,j,v_1,\dots,v_i) \in \mathcal{C}_i} \frac{g_{jk} \sigma^k}{k! v_1! \dots v_i!} \prod_{1 \leq p \leq i} \left( \hat{x}_{t-1}^{(p)} \right)^{v_p} \quad (30)$$

where

$$\mathcal{C}_i = \left\{ (k, j, v_1, \dots, v_i) \mid k + \sum_{p=1}^i p v_p = i, \sum_{p=1}^i v_p = j \right\}$$

and where

$$g_{j,k} \equiv \frac{\partial^{j+k} g(x_{t-1}, \sigma)}{\partial x_{t-1}^j \partial \sigma^k} \Big|_{(\bar{x}, 0)}$$

**Proof:** The  $m$ -th order Taylor expansion for  $x_t = f_t(\sigma)$  is

$$x_t \approx \sum_{i=1}^m \frac{1}{i!} x_t^{(i)} \sigma^i \quad (31)$$

or

$$x_t \approx \sum_{i=1}^m \hat{x}_t^{(i)}$$

where

$$\hat{x}_t^{(i)} = \frac{1}{i!} x_t^{(i)} \sigma^i$$

The  $m$ -th order Taylor expansion of  $g$  delivers

$$x_t \approx \sigma \epsilon_t + \sum_{j=0}^m \sum_{k=0}^{m-j} \frac{(j+k)!}{j! k! (j+k)!} g_{j,k} x_{t-1}^j \sigma^k$$

where we note that  $g_{0,0} = 0$  per our assumption that  $\bar{x} = 0$ . Replace  $x_{t-1}$  on the right hand side with the  $m$ -th order Taylor expansion (31), lagged by

one period, and exploit the usual multinomial formulas to calculate powers of sums to obtain

$$\begin{aligned}
x_t &\approx \sigma \epsilon_t + \sum_{j=0}^m \sum_{k=0}^{m-j} \frac{\frac{(j+k)!}{j!k!}}{(j+k)!} g_{j,k} \left( \sum_{p=1}^m \frac{1}{p!} x_{t-1}^{(p)} \sigma^p \right)^j \sigma^k \\
&\approx \sigma \epsilon_t + \sum_{j=0}^m \sum_{k=0}^{m-j} \sum_{v_1+\dots+v_m=j} g_{j,k} \frac{\frac{(j+k)!}{j!k!}}{(j+k)!} \frac{j!}{v_1! \dots v_m!} \prod_{1 \leq p \leq m} \left( \frac{1}{p!} x_{t-1}^{(p)} \sigma^p \right)^{v_p}
\end{aligned}$$

Sorting coefficients and then comparing the coefficients for  $\sigma^1$  to  $\sigma^m$  to the coefficients in (31) delivers the result (where it may be useful to note that the formula above also contains terms involving powers of  $\sigma$  higher than  $m$ : we do not compare those). In particular, any combination of coefficients  $(k, j, v_1, \dots, v_m)$  with the property that  $k + \sum_{p=1}^m p v_p = i$  delivers a term for  $\sigma^i$ . Note, that therefore  $v_{i+1} = 0, \dots, v_m = 0$ : we only need to specify  $v_1, \dots, v_i$ . With that, we obtain the set  $\mathcal{C}_i$ . •

It is instructive to explicitly write out the expansion up to order four. Taking into account that  $g_{1,0} = g_\sigma(\bar{x}; 0) = 0$  and  $g_{1,1} = g_{x\sigma}(\bar{x}; 0) = 0$ , the expressions for  $\hat{x}_t^{(1)}$  and  $\hat{x}_t^{(2)}$  are given by (25) and (26), i.e., they deliver the usual second-order pruning as shown above. For the third expansion term, we obtain

$$\begin{aligned}
\hat{x}_t^{(3)} &= \frac{g_{(0,3)} \sigma^3}{6} + \frac{\hat{x}_{t-1}^{(1)} g_{(1,2)} \sigma^2}{2} + \frac{(\hat{x}_{t-1}^{(1)})^2 g_{(2,1)} \sigma}{2} + \hat{x}_{t-1}^{(2)} g_{(1,1)} \sigma \\
&+ \hat{x}_{t-1}^{(3)} g_{(1,0)} + \hat{x}_{t-1}^{(1)} \hat{x}_{t-1}^{(2)} g_{(2,0)} + \frac{(\hat{x}_{t-1}^{(1)})^3 g_{(3,0)}}{6} \tag{32}
\end{aligned}$$

For the fourth term, we obtain

$$\begin{aligned}
\hat{x}_t^{(4)} &= \frac{g_{(0,4)}\sigma^4}{24} + \frac{\hat{x}_{t-1}^{(1)}g_{(1,3)}\sigma^3}{6} + \frac{\hat{x}_{t-1}^{(2)}g_{(1,2)}\sigma^2}{2} + \frac{(\hat{x}_{t-1}^{(1)})^2g_{(2,2)}\sigma^2}{4} \\
&+ \hat{x}_{t-1}^{(3)}g_{(1,1)}\sigma + \hat{x}_{t-1}^{(1)}\hat{x}_{t-1}^{(2)}g_{(2,1)}\sigma + \frac{(\hat{x}_{t-1}^{(1)})^3g_{(3,1)}\sigma}{6} + \hat{x}_{t-1}^{(4)}g_{(1,0)} \\
&+ \left( \hat{x}_{t-1}^{(1)}\hat{x}_{t-1}^{(3)} + \frac{(\hat{x}_{t-1}^{(2)})^2}{2} \right) g_{(2,0)} + \frac{(\hat{x}_{t-1}^{(1)})^2\hat{x}_{t-1}^{(2)}g_{(3,0)}}{2} + \frac{(\hat{x}_{t-1}^{(1)})^4g_{(4,0)}}{24}
\end{aligned} \tag{33}$$

The formulas we have provided deliver the additional terms for pruning of higher order. Pruning to second order is

$$x_t \approx \hat{x}_t^{(1)} + \hat{x}_t^{(2)},$$

pruning to third order is

$$x_t \approx \hat{x}_t^{(1)} + \hat{x}_t^{(2)} + \hat{x}_t^{(3)},$$

pruning to fourth order is

$$x_t \approx \hat{x}_t^{(1)} + \hat{x}_t^{(2)} + \hat{x}_t^{(3)} + \hat{x}_t^{(4)}$$

and so forth.

While (32) may feel like a rather natural extension of (26), and while the literature has already suggested this particular third-order pruning scheme, we find it less plausible that the specifics of equation (33) are easy to guess. It naturally arises out of our algorithm, however. A key contribution of this paper is to solidify the appropriate third-order pruning scheme, to provide this fourth-order pruning scheme, and to provide an algorithm for computing schemes of any order.

**Corollary 1** *Suppose that  $\mathbf{x}_t \in \mathbf{R}^n$  and  $\epsilon_t \in \mathbf{R}^n$  rather than  $\mathbf{R}$ . Then, replace (34) in proposition 3 by*

$$\hat{\mathbf{x}}_t^{(i)} = 1_{i=1} \sigma \epsilon_t + \sum_{(k,j,v_1,\dots,v_i) \in \mathcal{C}_i} \frac{g_{jk}\sigma^k}{k!v_1! \dots v_i!} \underbrace{\left( \hat{\mathbf{x}}_t^{(1)} \right)^{\otimes v_1} \otimes \dots \otimes \left( \hat{\mathbf{x}}_t^{(i)} \right)^{\otimes v_i}}_i \tag{34}$$

where, as customary,

$$\mathbf{x}_t^{\otimes n} \equiv \underbrace{\mathbf{x}_t \otimes \dots \otimes \mathbf{x}_t}_n$$

and where  $g_{jk}$  is a matrix, with each row collecting the  $n^j$  partial derivatives of order  $k$  with respect to  $\sigma$  and of total order  $j$  with respect to entries in  $\mathbf{x}_{t-1}$  sorted in the same way as  $\mathbf{x}_t^{\otimes n}$ .

**Proof:** The extension to the multivariate case is straightforward. •

## 6 Relationship to Judd (1998)

The expressions of proposition 1 and their logic turn out to be close to the derivations provided by section 13.3 in Judd (1998). There, Judd shows how to do a Taylor expansion for an continuous-time initial value problem indexed by a parameter  $\sigma$  (which he denotes by  $\epsilon$ : we switch the notation for better comparison). The initial value problem in his equation (13.3.1) is

$$\dot{x} \equiv \frac{\partial x(t; \sigma)}{\partial t} = g(x, t; \sigma) \quad (35)$$

Like us, he supposes that an exact solution as well as all derivatives can be obtained at  $\sigma = 0$ , allowing for the Taylor series in equation (13.3.3) or

$$x(t; \sigma) \approx \sum_{k=0}^n a_k(t) \frac{\sigma^k}{k!} \quad (36)$$

He shows that

$$\dot{a}_1(t) = g_x a_1 + g_\sigma \quad (37)$$

$$\dot{a}_2(t) = g_x a_2 + g_{xx} a_1^2 + g_{x\sigma} a_1 + g_{\sigma\sigma} \quad (38)$$

see his equations (13.3.4) and (13.3.5). Comparing (37) and  $a_1(t), a_2(t)$  to equations (21) and  $x_t^{(1)}, x_t^{(2)}$  shows a remarkable degree of similarity. As Judd remarks in his section, it is not hard to generalize his formulas to the discrete-time case. In essence here, stochastic terms as well as a treatment of the initial condition get introduced as well, but we do not wish to exaggerate the differences. Indeed, one might come to the conclusion that pruning has already been developed in Judd (1998), and that the formulas in KKSS are just versions of the formulas in Judd (1998), as we have shown above.

Of course, Judd (1998) also derives Taylor expansions of the value function as well as of the policy function in his chapter 13.5, which are functions of the state, and give rise to the non-pruned solutions. Judd's book provides solutions to well-posed questions, and that is its purpose. However, it does not seem to highlight the issue that for the objective of simulation, the approach in section 13.3 is more appropriate than the approach in section 13.5. This is not meant to take away from his contribution, but to put it into context. KKSS deserve credit for raising awareness, that issues arise for the objective of simulation, if a solution is obtained from a higher-order Taylor expansion of the policy function, and for proposing pruning as a work-around.

Our paper finally connects these two perspectives. To our knowledge, this has not been done before. In essence, we point out that the answer to the explosiveness issue raised in KKSS can be solved by posing the appropriate problem, namely, how to find a path for the variables of interest approximating the true path, per Taylor expansion around the path given by the constant steady state. Once the problem is posed in this manner, standard Taylor expansion or a version of Judd (1998), section 13.3 provide the answer.

## 7 Conclusions

Kim-Kim-Schaumburg-Sims (2008) have proposed pruning to deal with the challenge of finding stationary simulations, when utilizing a second order approximation for the recursive law of motion. In this paper, we have provided a theory of pruning. More specifically, we have shown that pruning can be understood as a standard Taylor approximation, when stating the variable of interest at date  $t$  as a function of the standard deviation parameter  $\sigma$  as well as variables which are invariant to it. A key contribution of this paper is to solidify the appropriate third-order pruning scheme, to provide the forth-order pruning scheme, and to provide an algorithm for computing schemes of any order. We have established the connection between pruning and the approach provided by Judd (1998) on perturbing dynamical systems.

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